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C. K. OGDEN

HARMONICS,

O R

THE PHILOSOPHY OF MUSICAL SOUNDS.

B Y

ROBERT SMITH, D.D, F.R.S;
And Master of Trinity College
In the University of Cambridge.

O Decus Phæbi —————

————— *ô laborum*

Dulce lenimen.

THE SECOND EDITION,

Much improved and augmented.

L O N D O N :

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MDCCCLIX.

Scimus musicen, matheſin, atque adeo veram phyſicam
noſtris moribus NON abeſſe à Principis perſona: Quæ
quidem omnia apud Græcos non laudè ſolùm, ſed honore
et gloria digna ducebantur.

Epaminondas, Imperator ille inſignis, ne dicam ſum-
mus vir unus omnis Græciæ, philoſophiam et muſicam
egregie didicit. Nam doctus eſt à Dionyſio, qui fuit
eximia in muſicis gloria. At philoſophiæ præceptorem
habuit Lyſin Pythagoreum, neque prius eum à ſe dimiſit,
quam doctrinis tanto antecceſſit alios, ut facilè intelligi
poſſet, pari modo ſuperaturum omnes in cæteris artibus.

Corn. Nep. vit. Epam. ſub initio.

T O
HIS ROYAL HIGHNESS
W I L L I A M
DUKE OF CUMBERLAND,

This Philosophical Treatise,
For a lasting Testimony of Gratitude,
Is humbly offered and dedicated,

By His ROYAL HIGHNESS'S

most devoted and

most dutiful servant

ROBERT SMITH

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T H E P R E F A C E

TO THE FIRST EDITION.

THE want of an elementary treatise of harmonics, such as might properly have been quoted in support of my demonstrations, has obliged me to begin the following work from the first principles of the science.

The antient theorists considered no other consonances than such as are perfect, and yet all their musical scales composed of these consonances, have in practice been found disagreeable. The reason is, they necessarily contain some imperfect concords, whose imperfections are too gross for the ear to bear with.

The skill of the moderns has been chiefly employed in the business of tempering the antient scales, that is, in distributing those grosser imperfections in some of the concords, among all the rest or the greater part of them. By

which means, though the number of imperfect concords be greatly increased, yet if their several imperfections be but as much diminished, the ear will be less offended than before. Because it is the transition from a better harmony to a worse, which chiefly gives the offence; as is evident to any one that attends to a piece of music performed upon an instrument badly tuned. It follows then that the instrument would be better in tune, if all the consonances were made as equally harmonious as possible, though none of them were perfect.

And if this be the true design in tuning an instrument, or tempering a scale of sounds, a theorist ought to begin with the simplest case; and inquire in the first place, whether it be possible for two imperfect consonances to be made equally harmonious; and if so, what must be the proportion of their temperaments or imperfections; and also whether different consonances require different proportions. These and the
the

the like questions being rightly settled, we may then determine in what proportion those grosser imperfections in the antient scales ought to be distributed, so as to make all the concords equally harmonious in their kind, either exactly or as near as possible.

But as none of the writers that I have seen, have attempted to give us the least notion of the nature and constitution of imperfect consonances, nor of any one property or proportion of their effects upon the ear, except a single conjecture whose contrary is true (*a*), it was not possible for them to determine, from the principles of science, what distribution of those grosser imperfections in the antient systems, would produce the most harmonious scale of musical sounds.

As this is one of the most difficult and important problems in harmonics, in order to a scientific solution of it I found it necessary to premise a Theory of Imperfect Consonances (*b*), wherein

a 4

I

(*a*) Prop. XIII. coroll. 8.

(*b*) Sect. vi.

I have demonstrated as many properties of their Periods, Beats and Harmony as I judged sufficient for solving that problem, and probably any other that belongs to harmonics. This theory with its preliminaries and consequences takes up a large part of the present treatise. As to the rest I chuse to refer the reader to the book it self or the Index, rather than trouble him with a further account of it: a short one would be imperfect or obscure, and a perfect one, too long for a preface.

Having been asked more than once, whether an ear for music be necessary to understand harmonics, it may not be amiss to give this answer: That a musical ear is not necessary to understand the philosophy of musical sounds; no more than the eye, to understand that of colours. Our late Professor of Mathematics was an instance of the latter case, and the xxth proposition of this treatise affords an instance of the former. For by the solution of that proposition and a new way of tuning an
viiith,

viiith, described in prop. xi, schol. 2, art. 6, a person of no ear at all for music may soon learn to tune an organ according to any proposed temperament of the scale; and to any desired degree of exactness, far beyond what the finest ear unassisted by theory can possibly attain to: and the same person, if he pleases, may also learn the reason of the practice.

But though an ear for music is not necessary to understand this treatise, yet those that are acquainted with musical sounds will more readily apprehend many parts of it, and receive more pleasure from them.

In the first scholium to prop. xx, I observed that the winter season had prevented me from tuning an organ by the second table of beats, in order to try what effect the system of Equal Harmony might have upon the ear. But upon telling Mr. *Turner*, one of our organists at Cambridge, how he might approach near enough to that system, by flattening the major iii^{ds},
till

till the beats of the vth and vith major with the same base, went equally slow, by his great dexterity and skill in tuning he presently put my rule in execution upon a stop of his organ; and affirmed to me, he never heard so fine harmony before, especially in the flat keys; but he added, that for want of more sounds in every octave the false concords were more intolerable than ever: and no wonder, as their common difference from true concords was then increased from one fifth to one fourth of the tone.

Nor will it be improper to mention a like experiment made by the accurate hand of Mr. *Harrison*, well known to the curious in mechanics by his admirable inventions in watch-work and clock-work for keeping time exactly both at sea and land: which if duly encouraged and pursued will undoubtedly prove of excellent use in navigation; by correcting the sea-charts, with respect to longitude, as well as the reckonings of a ship, to as great
ex-

exactness, in all probability, as need be desired.

But in regard to the experiment I was going to mention, he told me he took a thin ruler equal in length to the smallest string of his Base Viol, and divided the ruler as a monochord, by taking the interval of the major III^d , to that of the VIII^{th} , as the diameter of a circle, to its circumference. Then by the divisions on the ruler applied to that string, he adjusted the frets upon the neck of the viol, and found the harmony of the consonances so extremely fine, that after a very small and gradual lengthening of the other strings, at the nut, by reason of their greater stiffness, he perfectly acquiesced in that manner of placing the frets.

It follows from Mr. *Harrison's* assumption, that his III^d major is tempered flat by a full fifth of a comma. My III^d determined by theory, upon the principle of making all the concords within the extent of every three octaves as equally harmonious as possible, is
tem-

tempered flat by one ninth of a comma; or almost one eighth, when no more concords are taken into the calculation than what are contained within one octave. That theory is therefore supported on one hand by Mr. *Harrison's* experiment, and on the other by the common practice of musicians, who make the major III^d either perfect or generally sharper than perfect.

We may gather from the construction of the Base Viol, that Mr. *Harrison* attended chiefly, if not solely to the harmony of the consonances contained within one octave; in which case the differences between his and my temperaments of the major III^d, VIth and Vth, and their several dependents, are respectively no greater than 4, 3 and 1 fiftieth parts of a comma. And considering that any assigned differences in the temperaments of a system, will have the least effect in altering the harmony of the whole when at the best, I think a nearer agreement of that experiment
with

with the theory could not be reasonably expected.

Upon asking him why he took the interval of the major III^d to that of the VIII^{th} as the diameter to the circumference of a circle, he answered, that a gentleman lately deceased had told him it would bring out a very good division of a monochord. Whoever was the author of that hypothesis, for so it must be called, as having no connexion with any known property of sounds, he took the hint, no doubt, from observing that as the octave, consisting of five mean tones and two limmas, is a little bigger than six such tones, or three perfect major III^{ds} , so the circumference of a circle is a little bigger than three of its diameters.

When the monochord was divided upon the principle of making the major III^d perfect, or but very little sharper, as in Mr. *Huygens's* system resulting from the octave divided into 31 equal intervals, Mr. *Harrison* told me the major VI^{ths} were very bad, and much worse

worse than the vth. In which he judged rightly, as I further satisfied my self by trying the experiment upon an organ; and being solicitous to know the reason of that effect, that is, why the vth and vith major, when equally tempered, should differ so in their harmony, after various attempts I satisfied my curiosity.

With a view to some other inquiries I will conclude with the following observation. That, as almost all sorts of substances are perpetually subject to very minute vibrating motions, and all our senses and faculties seem chiefly to depend upon such motions excited in the proper organs, either by outward objects or the power of the Will, there is reason to expect, that the theory of vibrations here given will not prove useless in promoting the philosophy of other things besides musical sounds.

ROB. SMITH.

Trinity College,
Cambridge, Dec. 31. 1748.

THE PREFACE

TO THE SECOND EDITION.

*I*N this second edition of these harmonics, besides many smaller improvements, the properties of the periods, beats and harmony of imperfect consonances are more explicitly demonstrated (a) and confirmed by very easy experiments (b). The ultimate ratios of the periods and beats, which are generally more useful and elegant than the exact ratios, are proved to be sufficiently accurate for most purposes in harmonics (c). More methods are added for finding the pitch of an organ (d) and for tuning it, either by estimation and judgment of the ear (e), or more exactly and readily by isochronous beats of different concords (f), as well as by
complete

(a) Lemma to prop. ix, and prop. ix, xi and corollaries.

(b) Prop. xi. schol. 2.

(c) Prop. xi. schol. 1.

(d) Prop. xviii. and schol. &c.

(e) Sect. ix. art. 1.

(f) Prop. xx. schol. 2.

complete tables of beats. An enquiry is made whether coincident pulses be necessary, or only accidental to a perfect consonance (g).

And lastly, as the harpsichord has neither strings nor keys for any of these sounds D, A*, E*, B*, F**, A^b, D^b, G^b, &c, which yet are so often wanted that far the greater part of the best compositions cannot be performed without them, except by substituting for them E^b, B^b, F, C, G, G*, C*, F*, &c, respectively, which by differing from them by near a fifth part of the tone, make very bad harmony; and as the old expedient for introducing some of those sounds by inserting more keys in every octave, is quite laid aside by reason of the difficulty in playing upon them; I have therefore invented a better expedient, by causing the several keys of those substitutes, E^b, B^b, F, C, G, G*, C*, F*, &c, to strike either E^b or D*, B^b or A*, F or E*, C or B*,*

G

(g) Prop. xi. schol. 4. art. 7. &c.

G or F^{**}, G^{*} or A^b, C^{*} or D^b, F^{*} or G^b, &c.

For since both the sounds in any one of those couples are seldom or never used in any one piece of music, the musician by moving a few stops before he begins to play it, can immediately introduce that sound in each couple, which he foresees is either always or ofteneſt used in the piece before him.

Two different conſtruction of thoſe ſtops are here deſcribed (b), one of which is applicable at a ſmall expenſe to any harpſichord ready made, and the other to a new harpſichord, and upon putting them both in practice, they have perfectly answered my expectation.

Several properties and advantages of this changeable ſcale are deſcribed in the eighth Section. In a word, the very worſt keys in the common defective ſcale, by changing a few ſounds are preſently made as complete as the beſt in that ſcale, and more harmonious too, becauſe the

a change-

(b) Sect. VIII. art. 18, 19.

changeable scale admits of the very best temperament, and, which is another advantage, will therefore stand longer in tune than the common scale which cannot admit that temperament.

These improvements of the harpsichord, it is hoped, may encourage others to apply the like methods to the scale of the organ, which is equally capable of them and to greater advantages.

ROB. SMITH.

Trinity College, Cambridge,
Octob. 21, 1758.

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ADVERTISEMENT.

A Complete System of Optics in four Books, viz. a popular, a mathematical, a mechanical and a philosophical treatise, by Dr. Smith: Cambridge 1738, 2 Vol. 4^{to}.

Harmonia Mensurarum, five analysis et synthesis per rationum et angulorum Mensuras promotæ: accedunt alia opuscula, nempe de Limitibus errorum in mixta mathesi, de methodo Differentiarum Newtoniana, de constructione Tabularum per differentias, de descensu Gravium, de motu Pendulorum in cycloide et de motu Projectilium, per Rôgerum Cotesium: Edidit et auxit Robertus Smith: Cantabrigiæ 1722, 4^{to}.

Hydrostatical and Pneumatical Lectures by Mr. Cotes, published by Dr. Smith: Cambridge 1747, 2^d Edition, 8^{vo}.

HARMONICS.

SECTION I.

Philosophical Principles of Harmonics.

1. **SOUND** is caused by the vibrations of elastic bodies, which communicate the like vibrations to the air, and these the like again to our organs of hearing.

Philosophers are agreed in this, because sounding bodies communicate tremors to distant bodies. For instance, the vibrating motion of a musical string puts others in motion, whose tension and quantity of matter dispose their vibrations to keep time with the pulses of air, propagated from the string that was struck. *Galileo* explains this phenomenon by observing, that a heavy pendulum may be put in motion by the least breath of the mouth, provided the blasts be often repeated and keep time exactly with the vibrations of the pendulum; and also by the like art in raising a large bell; and probably he was the first that rightly explained that phenomenon (*a*).

2. If

(*a*) For he says, in the person of another, il problema poi trito delle due corde tefe all'unifono, che al suono dell'

A

una

2. If the vibrations be isochronous the sound is called Musical, and is said to continue at the same Pitch ; and to be Acuter, Sharper or Higher than any other sound whose vibrations are slower ; and Graver, Flatter or Lower (*b*) than any other whose vibrations are quicker.

For while a musical string vibrates, if its tension be increased or its length be diminished, its vibrations will be accelerated ; and experience shews that its sound is altered from what is called a graver to an acuter ; and on the contrary. And the like alteration of the pitch of the sound will follow, when the same tension is given by a weight, first to a thicker or a heavier string, and after that to a smaller or a lighter of the same length, as having less matter to be moved by the same

una l'altra si muova et attualmente risuona, mi resta ancora irrisolto ; come anco non ben chiare le forme delle consonanze et altre particolarità. *Dialogo 1° attenente alla Meccanica*, towards the end.

(*b*) As the ideas of acute and high, grave and low, have in nature no necessary connexion, it has happened accordingly, as Dr. Gregory has observed in the preface to his edition of *Euclid's* works, that the more antient of the Greek Writers looked upon grave sounds as high, and acute ones as low, and that this connexion was afterwards changed to the contrary by the less antient Greeks, and has since prevailed universally. Probably this latter connexion took its rise from the formation of the voice in singing, which *Aristides Quintilianus* thus describes. Γίνε-
ται δὲ ἡ μὲν βαρύτης, κάτωθεν ἀναφερόμενος τῷ πνεύ-
ματι, ἡ δὲ ὀξύτης, ἐπιπολῆς προϊέμενος. p. 8. Et qui-
dem gravitas fit, si ex inferiore parte (gutturis) spiritus sur-
sum feratur, acumen vero, si per summam partem prorum-
pat, as *Meibomius* translates it in his notes. pag. 208.

same force of tension. And these changes in the pitch of the sound are found to be constantly greater or lesser, according as the length, tension, thickness or density of the string is more or less altered (*c*).

3. Therefore if several strings, however different in length, thickness, density and tension, or other sounding bodies vibrate all together in equal times, their sounds will all have one and the same pitch, however they may differ in loudness or other qualities, and are therefore called Unisons: and on the contrary, the vibrations of unisons are isochronous.

This observation reduces the theory of all sorts of musical sounds to that of the sounds of a single string; I mean with respect to their gravity and acuteness, which is the principal subject of Harmonics (*d*).

4. Con-

(*c*) The Greek musicians rightly describe the difference between the manner of singing and talking. They considered two motions in the voice, κινήσεις δύο; the one continued and used in talking, ἡ μὲν συνεχὴς τε καὶ λογική, the other discrete and used in singing, ἡ δ' ἐστὶ διασηματική τε καὶ μελωδική. In the continued motion, the voice never rests at any certain pitch, but waves up and down by insensible degrees; and in the discrete motion it does the contrary; frequently resting or staying at certain places, and leaping from one to another by sensible intervals: *Euclid's* Introductio Harmonica, p. 2. I need not observe, that in the former case, the vibrations of the air are continually accelerated and retarded by turns and by very small degrees, and in the latter by large ones.

(*d*) *Ptolemy* says, Ἀρμονικὴ μὲν ἐστὶ δύναμις ἀλλοτρίᾳ τῶν ἐν τοῖς ψόφοις, πρὸς τὸ ὅζον καὶ βαρὺν, διαφορῶν.

4. Consequently the wider and narrower vibrations of a musical string, or of any other body sounding musically, are all isochronous very nearly.

Otherwise, while the vibrations decrease in breadth till they cease, the pitch of the sound could not continue the same; as by the judgment of the ear we perceive it does, if the first vibrations be not too large: in which case the sound is a little acuter at the beginning than afterwards.

5. In like manner, since the pitch of the sound of a string or bell or other vibrating body, does not alter sensibly while the hearer varies his distance from it; it follows that the larger and lesser vibrations of the particles of air, at smaller and greater distances from the sounding body, are all isochronous: and consequently that the little spaces described by the vibrating particles are every where proportional to the celerity and force of their motions, as in a pendulum (*e*). And this difference of force, at different distances from the sounding body, causes a difference in the loudness of the sound, but not in its pitch.

6. It follows also, that the harmony of two or more sounds, according as it is perfect or imperfect when heard at any one distance, will also be perfect or imperfect at any other distance: which

Harmonics is a power apprehending the differences of sounds, with respect to gravity and acuteness.

(*e*) See Newton's Princip. Lib. II. Prop. 47.

which being a known fact in a ring of bells for instance, is mentioned here as a confirmation of these principles of Harmonics.

7. If two musical strings have the same thickness, density and tension, and differ in length only, (which for the future I shall always suppose,) mathematicians have demonstrated, that the times of their single vibrations are proportional to their lengths (*f*).

8. Hence if a string of a musical instrument be stopt in the middle, and the sound of the half be compared with the sound of the whole, we may acquire the idea of the interval of two sounds, whose single vibrations (always meaning the times) are in the ratio of 1 to 2; and by comparing the sounds of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{8}{9}$, $\frac{9}{10}$, &c. of the string with the sound of the whole, we may acquire the ideas of the intervals of two sounds, whose single vibrations are in the ratio of 2 to 3, 3 to 4, 3 to 5, 4 to 5, 5 to 6, 8 to 9, 9 to 10, &c.

9. A Musical Interval is a quantity of a certain kind (*g*), terminated by a graver and an acuter sound.

In

(*f*) As a clear and exact demonstration of this curious Theorem depends upon one or two more, of no small use in Harmonics, and requires a little of the finer sort of geometry, which cannot well be applied in few words, I have therefore reserved it to the last Section of this Treatise; which the reader may consult, or, taking it for granted at present, may proceed without interruption; as he likes best:

(*g*) See Dr. Wallis's preface to *Porphyry's* comment on *Ptolemy's* Harmonics. Oper. Math. vol. III. *Euclid* says,

In a ring of bells, for example, the sounds of the first and second bells, counting either from the biggest or the least, terminate a certain interval; those of the first and third a greater interval; those of the first and fourth a greater still; &c. So that the interval increases by degrees, either as the graver of the two sounds descends, or as the acuter ascends; and within the interval of the sounds of the biggest and least bells, the intervals between the sounds of all the rest are contained.

10. Musical intervals are Measures of the Ratios of the times of the single vibrations of the terminating sounds, or, *cæteris paribus*, of the lengths of the sounding strings (*b*).

For it is observable in the experiments last mentioned (*i*) and is universally allowed by musicians, that when the lengths of those strings have

an Interval is τὸ περιεχόμενον ὑπὸ δύο φθόγων ἀνωμοίων ὀξύτητι καὶ βαρύτητι, what is contained by two sounds different in gravity and acuteness. *Introductio Harmonica* p. 1. *Aristoxenus* defines a musical sound thus, φωνῆς πλῆσις ἐπὶ μίαν τάσιν ὁ φθόγος, sonus est vocis casus in unam tensionem; and an interval thus, Διάστημα δ' ἐστὶ τὸ ὑπὸ δύο φθόγων ὀρισμένον, μὴ τὴν αὐτὴν τάσιν ἔχοντων; intervallum vero est, quod duobus sonis, non eandem tensionem habentibus, terminatur. And he adds, that it is τόπος δ' ἐκλιπὸς φθόγων, ὑψιτέρων μὲν τῆς βαρυτέρως τῶν ὀξύτεσσιν τὸ διάστημα τάσεων, βαρυτέρων δ' ἐ τῆς ὀξύτερος; a place capable of sounds, that are acuter than the graver of the two tensions (tones or sounds) that terminate the interval, and graver than the acuter of them. pag. 15.

(*b*) Article 7.

(*i*) Art. 8.

have the same ratio, the interval of their sounds is the same, whatever be their pitch; that if the acuter of the two sounds be raised higher, and consequently the ratio of the lengths of those strings be increased, the interval is increased; and on the contrary, if the acuter sound be depressed lower, that the said ratio and interval are diminished, and reduced to nothing when the strings have the ratio of equality whose magnitude is nothing.

Plate I. Fig. 1. Now let the times of the single vibrations of the strings *A*, *B*, *C*, *D*, &c, be continual proportionals in any ratio. Then since the interval of the sounds of *A* and *B* is equal to that of *B* and *C*, or of *C* and *D*, &c, by adding equal intervals together and equal ratios together, it follows, that the interval of the sounds of *A* and *C*, whose ratio is duplicate of *A* to *B* or of *B* to *C*, is double the interval of the sounds of *A* and *B*, or of *B* and *C*; and that the interval of the sounds of *A* and *D*, whose ratio is triplicate of *A* to *B*, is also triple the interval of the sounds of *A* and *B*, or of *B* and *C* or of *C* and *D*. So that the interval of the sounds of *A* and *C*, is to that of *A* and *D*, as 2 to 3; and the like is evident of any other equimultiples of the proposed ratios and intervals, whatever be their number and magnitude.

II. Therefore musical intervals are proportional to the logarithms of the ratios of the single

A 4

vibra-

vibrations of the terminating sounds, or, *cæteris paribus*, of the lengths of the vibrating strings. Because logarithms are numeral measures of ratios; and all sorts of measures, of the same magnitudes are proportional to one another (*k*).

12. For brevity sake the word vibration is often used for the time of a complete vibration, which passes between the departure of the vibrating body from any assigned place and its return to the same. Such is the time between the successive pulses of air upon the ear; a pulse being made while the air is compressed and condensed in its progress, but not in its regress; it being then relaxed and rarified to a greater degree than the quiescent air is (*l*). And though the pulses of sounds of a different pitch have different durations, they may yet be abstractly considered as if they were instantaneous; by taking only the middle instant of each pulse.

(*k*) See Mr. Cotes's *Harmonia Mensurarum*, pag. 1.

(*l*) See *Newton's Principia*, Book 2. Prop. 43. Cor. 1.

SECTION

SECTION II.

Of the Names and Notation of consonances and their intervals.

I. **P**LATE I. Fig. 2. If a musical string CO and its parts DO , EO , FO , GO , AO , BO , cO , be in proportion to one another as the numbers 1 , $\frac{8}{9}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{8}{15}$, $\frac{1}{2}$, their vibrations will exhibit the system of 8 sounds which musicians denote by the letters C , D , E , F , G , A , B , c .

Fig. 3. And supposing those strings to be ranged like ordinates to a right line Cc , and their distances CD , DE , EF , FG , GA , AB , BC , not to be the differences of their lengths, as in fig. 2, but to be of any magnitudes proportional to the intervals of their sounds, the received Names of these intervals are shewn in the following Table; and are taken from the numbers of the strings or sounds in each interval inclusively; as a Second, Third, Fourth, Fifth, &c, with the epithet of *major* or *minor*, according as the name or number belongs to a greater or smaller total interval; the difference of which results chiefly from the different magnitudes of the major and minor second, called the Tone and Hemitone.

C..

$$C \dots D \dots E \dots F \dots G \dots A \dots B \dots c \dots$$

$$1 \dots \frac{8}{9} \dots \frac{4}{5} \dots \frac{3}{4} \dots \frac{2}{3} \dots \frac{3}{5} \dots \frac{8}{15} \dots \frac{1}{2} \dots$$

Perfect Ratios, Interval's Names, Marks, Elements.

$C:c :: 2:1$	Cc	Octave	VIII	$3T + 2t + 2H$
$B:c :: 16:15$	Bc	Hemitone	H or 2 ^d	
$C:B :: 15:8$	CB	VII major	VII	$3T + 2t + H$
$C:D :: 9:8$	CD	Tone major	T or II	
$D:c :: 16:9$	Dc	7 th minor	7 th	$2T + 2t + 2H$
$A:c :: 6:5$	Ac	3 ^d minor	3 ^d	$T + H$
$C:A :: 5:3$	CA	VI major	VI	$2T + 2t + H$
$C:E :: 5:4$	CE	III major	III	$T + t$
$E:c :: 8:5$	Ec	6 th minor	6 th	$2T + t + 2H$
$G:c :: 4:3$	Gc	4 th minor	4 th	$T + t + H$
$C:G :: 3:2$	CG	V major	V	$2T + t + H$
$F:B :: 45:32$	FB	IV major	IV	$2T + t$
$B:f :: 64:45$	Bf	5 th minor	5 th	$T + t + 2H$
$D:E :: 10:9$	DE	Tone minor	t	
$81:80$		Comma	c	$T - t$

2. Hence it is, if the ratio of the single vibrations of any two sounds, or, *cæteris paribus*, of the lengths of two vibrating strings, be any of those in the first column of the table, that their interval, and the consonance too, retains the name in the third column, whether the intermediate sounds be present or absent.

3. Fig. 3. In the line Cc produced beyond c , if we take the intervals Dd , Ee , Ff , &c, severally

verally equal to the octave Cc , and make the length of the several strings at d, e, f , &c, equal to half the lengths of those at D, E, F , &c, all the intervals within this higher octave cc' , will also consist of major and minor tones and hemitones, ranged in the same order as in the lower octave Cc .

And the names of intervals larger than one or more octaves, are also taken from the number of the strings in them inclusively. Thus the interval Cd is called a Ninth, Ce a Tenth, Cf an Eleventh, Cg a Twelfth, &c, with the epithet of *major* or *minor* as before; and are thus denoted, IX or VIII + II, X or VIII + III, XIth or VIII + 4th, XII or VIII + V, &c, the units in the compound marks being constantly one more than those in the simple ones, because the intermediate string at the end of the octave is counted twice. The same is to be understood in all compounded notations.

4. A Comma is the interval of two sounds whose single vibrations have the ratio of 81 to 80, and is the difference of the major and minor tones (m).

5. Any one of the ratios in the first column of the foregoing Table, except 80 to 81, or any one of them compounded once or oftener with the ratio 2 to 1 or 1 to 2, is called a Perfect ratio when reduced to its least terms. And when the
times

(m) For the ratio of 9 to 8 diminished by the ratio of 10 to 9, is the ratio of 9×9 to 8×10 , or of 81 to 80.

times of the single vibrations of any two sounds have a perfect ratio, the consonance and its interval too is called Perfect; and is called Imperfect or Tempered when that perfect ratio and interval is a little increased or decreased.

6. Any small increment or decrement of a perfect interval is called respectively the Sharp or Flat Temperament of the imperfect consonance, and is measured most conveniently by the proportion it bears to a comma.

7. As the addition and subtraction of logarithms answers to the multiplication and division of their corresponding Tabular numbers, that is, to the composition and resolution of the ratios of those numbers to an unit; so the addition and subtraction of musical intervals answers to the composition and resolution of the ratios of the single vibrations of the terminating sounds, or, *cæteris paribus*, of the lengths of the vibrating strings: and on the contrary.

In the following examples, the composition and resolution of perfect ratios is intimated by the multiplication of their terms, placed, in the form of fractions, upright and inverted, respectively.

As

$$\left\{ \begin{array}{l} \text{As } \frac{1}{2} = \frac{8}{15} \times \frac{15}{16} = \frac{9}{16} \times \frac{8}{9} = \frac{3}{5} \times \frac{5}{6} = \\ \text{So VIII} = \text{VII} + 2^{\text{d}} = 7^{\text{th}} + \text{II} = \text{VI} + 3^{\text{d}} = \end{array} \right.$$

$$\frac{5}{8} \times \frac{4}{5} = \frac{2}{3} \times \frac{3}{4} = \frac{8}{9} \times \frac{8}{9} \times \frac{8}{9} \times \frac{9}{10} \times \frac{9}{10} \times \frac{15}{16} \times \frac{15}{16}.$$

$$6^{\text{th}} + \text{III} = \text{v} + 4^{\text{th}} = 3 \text{ T} + 2 \text{ t} + 2 \text{ H} \quad (n).$$

$$\left\{ \begin{array}{l} \text{As } \frac{3}{5} = \frac{3}{4} \times \frac{4}{5} \left| \frac{5}{8} = \frac{3}{4} \times \frac{5}{6} \right| \frac{2}{3} = \frac{4}{5} \times \frac{5}{6} \\ \text{So VI} = 4^{\text{th}} + \text{III} \left| 6^{\text{th}} = 4^{\text{th}} + 3^{\text{d}} \right| \text{v} = \text{III} + 3^{\text{d}} \end{array} \right.$$

$$\text{As } \frac{3}{4} = \frac{4}{5} \times \frac{15}{16} \left| \frac{3}{4} = \frac{5}{6} \times \frac{9}{10} \right| \frac{4}{5} = \frac{8}{9} \times \frac{9}{10}$$

$$\text{So } 4^{\text{th}} = \text{III} + 2^{\text{d}} \left| 4^{\text{th}} = 3^{\text{d}} + \text{t} \right| \text{III} = \text{T} + \text{t}$$

$$\left\{ \begin{array}{l} \text{As } \frac{5}{6} = \frac{8}{9} \times \frac{15}{16} \left| \frac{3}{5} \times \frac{3}{2} = \frac{9}{10} \right| \frac{2}{3} \times \frac{4}{3} = \frac{8}{9} \\ \text{So } 3^{\text{d}} = \text{T} + \text{H} \left| \text{VI} - \text{v} = \text{t} \right| \text{v} - 4^{\text{th}} = \text{T} \end{array} \right.$$

$$\text{As } \frac{3}{4} \times \frac{6}{5} = \frac{9}{10} \left| \frac{3}{4} \times \frac{5}{4} = \frac{15}{16} \right| \frac{8}{9} \times \frac{10}{9} = \frac{80}{81}$$

$$\text{So } 4^{\text{th}} - 3^{\text{d}} = \text{t} \left| 4^{\text{th}} - \text{III} = \text{H} \right| \text{T} - \text{t} = \text{c}.$$

8. Hence the hemitones, and tones major and minor, being the differences of the intervals, III, 4th, v, VI, and of their compliments to the octave, may be considered as the Elements that compound the intervals of all perfect concords,

(n) See the Column of Elements in the foregoing Table.

cords, as in the last column of the former Table compared with Fig. 3. So that the least intervals in a musical scale are founded upon the harmony of the concords (o).

SECTION III.

Of perfect consonances and the Order of their simplicity.

I. **P**LATE I. Fig. 4. When a single sound is heard, the series of equal times between the successive pulses of air that beat on the ear (*p*), may be represented by a series of equal parts contained in a right line; as in 02, 03, 04, &c. Consequently when two sounds are heard, two of those lines, as 02 and 03, will rightly represent the two series of equal times, if the magnitude of the equal parts in one line, be to the magnitude of those in the other, in the ratio of the single vibrations of the sounds: or, the whole lines being supposed equal, if the numbers of aliquot parts in each, as 2 and 3, be severally the same as the least numbers of the
vibra-

(o) The old method of resolving concords into their elements may be seen in Dr. *Wallis's* division of the monochord, or section of the musical Canon, as the antients called it. *Philosoph. Transact.* N^o. 238. or *Abridg.* by *Lowthorp.* vol. 1. p. 698. first edit.

(p) *Sect. I. Art. 12.*

vibrations of each sound, made in the same time represented by the line 02 or 03 (*q*).

2. And the sounds being heard together, if we conceive the two equal and parallel lines that rightly represent them, as 02 and 03, to coincide throughout, the points that divide the separate lines, will subdivide the combined lines into smaller portions, as in Fig. 5, representing a third series or Cycle of times, in which the pulses of both sounds interchangeably succeed one another in beating upon the ear.

3. Such a mixture of pulses, succeeding one another in a given cycle of Times, terminated at both ends by coincident pulses, and sufficiently repeated, is the physical cause that excites the sensation of a given consonance: Especially when considered as distinct from any other consonance, whose single vibrations having a different ratio from that of the former, will constitute a different cycle, and excite a different sensation. But if that ratio be the same, though the absolute times be different, the consonances are similar and may be looked upon as the same in this respect, that their cycles have the same form; the times in both having the same order, and the same proportions; and in this other also, that the interval of the sounds is the same (*r*).

4. This being premised, one consonance may be considered as more or less simple than another,

ac-

(*q*) See Art. 12. following.

(*r*) Art. 10. Sect. 1.

according as the cycle of times belonging to it, is more or less simple than the cycle belonging to the other. And upon this principle all consonances may be ranged in due Order of such simplicity, by the help of the following Rule.

5. *One Consonance is Simpler than another in the same Order, as the sum of the least terms, expressing the ratio of the single vibrations, is smaller than the like sum in the other consonance; and when several such sums are the same, these consonances are simpler in the same order, as the lesser terms of their ratios are smaller.*

For the simplicity of a consonance or cycle of times, consists partly in the number of times contained in the cycle, and partly in the different proportions they bear to one another.

Fig. 4. When the numbers of times in different cycles are different, and the times in each cycle are equal to one another, as when we combine the sounds 01 and 01, 01 and 02, 01 and 03, 01 and 04, 01 and 05, &c, the cycles of this sort may be ranged in the order of their simplicity above defined, by the order of the numbers of equal times in the cycles, or of the magnitudes of the numbers 1, 2, 3, 4, 5, 6, &c, or of 2, 3, 4, 5, 6, 7, &c, that is, of the sums of the terms of the ratios 1 to 1, 1 to 2, 1 to 3, 1 to 4, 1 to 5, &c.

In the other case, where the numbers of times in different cycles are the same, and the times in each cycle bear different proportions to one another, as when we combine the sounds 01 and 06,

06, 02 and 05, 03 and 04, that cycle is simpler than another, in which the equal times between the pulses of the acuter sound, are less interrupted and subdivided by the pulses of the graver.

Accordingly in the first of these cycles composed of 01 and 06, not one of the 6 equal times between the pulses of the acuter sound 06, is subdivided by any pulse of the graver 01; but in the second cycle composed of 02 and 05, one of the 5 equal times, between the pulses of the acuter sound 05, is subdivided by one pulse of the graver 02; and in the third cycle composed of 03 and 04, two of the 4 equal times in the acuter sound 04, are subdivided by 2 pulses of the graver 03. By which it appears, that the first cycle is simpler than the second, and the second simpler than the third; and that the order of simplicity of this sort of cycles, answers to the order of the magnitudes 1, 2, 3 of the lesser terms of the ratios.

6. Now by the first part of the rule above, the integers in the second column of the following table, are the several sums of the terms of the opposite ratios in the first, diminished by 1, which alters not the order of their magnitudes, but only makes the series begin with 1, answering to the simplest consonance.

By the second part of the rule, the ratios whose terms have the same sum, as 1 : 6, 2 : 5, 3 : 4, are ranged in the order of their lesser terms 1, 2, 3, or, which alters not the order, of those terms severally diminished by 1, as of 0, 1, 2, or of the

B

fractions

*A table of the Order of the simplicity
of consonances of two sounds.*

Ratios of the vibra- tions.	Order of the simpli- city.	Intervals of the found.	Continuation of the table.		
			1 : 15	15	3VIII + VII
1 : 1	1	0	1 : 16	16	4VIII
1 : 2	2	VIII	2 : 15	$16\frac{1}{8}$	2VIII + VII
1 : 3	3	VIII + V	5 : 12	$16\frac{1}{2}$	VIII + 3 ^d
			8 : 9	$16\frac{7}{8}$	T
1 : 4	4	2VIII	1 : 18	18	4VIII + T
2 : 3	$4\frac{1}{2}$	V	3 : 16	$18\frac{2}{9}$	2VIII + 4 th
1 : 5	5	2VIII + III	4 : 15	$18\frac{1}{3}$	VIII + VII
1 : 6	6	2VIII + V	9 : 10	$18\frac{8}{9}$	t
2 : 5	$6\frac{1}{3}$	VIII + III	1 : 20	20	4VIII + III
3 : 4	$6\frac{2}{3}$	4 th	5 : 16	$20\frac{2}{5}$	VIII + 6 th
1 : 7	7		1 : 22	22	
3 : 5	$7\frac{2}{3}$	VI	3 : 20	$22\frac{2}{11}$	2VIII + VI
1 : 8	8	3VIII	5 : 18	$22\frac{4}{9}$	VIII + 7 th
4 : 5	$8\frac{3}{4}$	III	8 : 15	$22\frac{7}{11}$	VII
1 : 9	9	3VIII + T	1 : 24	24	4VIII + V
1 : 10	10	3VIII + III	9 : 16	$4\frac{2}{3}$	7 th
2 : 9	$10\frac{1}{3}$	2VIII + T	1 : 28	28	
3 : 8	$10\frac{2}{3}$	VIII + 4 th	5 : 24	$28\frac{2}{7}$	2VIII + 3 ^d
5 : 6	$10\frac{4}{5}$	3 ^d	9 : 20	$28\frac{4}{7}$	VIII + t
1 : 12	12	3VIII + V	1 : 30	30	4VIII + VII
3 : 10	$12\frac{1}{3}$	VIII + VI	15 : 16	$30\frac{1}{15}$	H
4 : 9	$12\frac{1}{2}$	VIII + T			
5 : 8	$12\frac{2}{3}$	6 th	32 : 45	$76\frac{31}{8}$	IV
6 : 7	$12\frac{5}{6}$		45 : 64	$108\frac{22}{7}$	5 th

fractions $\frac{2}{3}$, $\frac{1}{3}$, $\frac{2}{3}$, whose common denominator 3 is the number of the ratios whose terms have the same sum 7. These fractions either by themselves or the mixt numbers 6, $6\frac{1}{3}$, $6\frac{2}{3}$, made by annexing them to the number 6, may therefore denote the order of the Modes of simplicity of such consonances as have the same Degree of simplicity denoted by 6 or 7—1. And thus the order of the simplicity of all consonances whatever, is denoted by the order of the magnitudes of the integers and mixt numbers in the second column of the table.

7. This series increases from unity in several arithmetical progressions, except that a term or two is here and there omitted, where ratios occur which being reducible to simpler terms, have been considered before, or else are not Perfect Ratios, which are such only whose terms are 1, 2, 3, 5, with their powers and products (s).

For example, writing down all the ratios in due order, whose terms make a given sum, as 1 to 8, 2 to 7, 3 to 6, 4 to 5, I reject the two middlemost for the reasons just mentioned, and place the rest in the first column of the table; which may thus be continued with certainty and order as far as we please.

8. Hence we may distinguish consonances into two sorts, Pure and Interrupted; pure, where none of the equal times between the pulses of the acuter sound, is subdivided by any interme-

B 2

diate

diat pulse of the graver ; and interrupted, when any of those equal times are interrupted by one or more pulses of the graver sound.

In the second column of the table, the least simple or lowest mode of each degree of interrupted consonancy, is every where placed above the next inferior degree of pure consonancy, as $4\frac{1}{2}$ above 5.

For should we depress the mode $4\frac{1}{2}$ to a place next below the degree 5, why not even to a place next below 6 ? though not below $6\frac{1}{3}$, as being a more complex mode of a less simple degree. But if that were allowable, by parity of reason we ought to depress $4\frac{1}{2}$, $6\frac{1}{3}$, $6\frac{2}{3}$, next below 7, though not below $7\frac{2}{3}$, and likewise $4\frac{1}{2}$, $6\frac{1}{3}$, $6\frac{2}{3}$, $7\frac{2}{3}$, below 8, though not below $8\frac{3}{4}$, and also $4\frac{1}{2}$, $6\frac{1}{3}$, $6\frac{2}{3}$, $7\frac{2}{3}$, $8\frac{3}{4}$ below 9 and 10, and so forth to infinity : which, by depressing all the modes of interrupted consonancy, below all the degrees of pure consonancy, would render them heterogeneal, and incapable of any order or comparison with one another. The table is therefore rightly ordered.

9. Hitherto we have only considered the number and proportions of the times in the cycle by which a consonance is represented, without regard to the quality of the pulses, as to magnitude, duration, strength, weakness or other accidents ; whereas the pulses of graver sounds are generally stronger, larger, obtuser, and of longer duration than those of acuter sounds, and affect the ear differently. But still this alters not the rational idea of the consonance, as above described, provided

provided we take the middle instant of each pulse as we did in Art. 1; nor does the ear perceive any alteration in the kind of mixture, or in the interval upon softening or swelling either sound, while the other retains the same strength.

10. It is well known in general, that simpler consonances affect the ear with a smoother and pleasanter sensation, and the less simple with a rougher and less pleasant one. And this analogy seems to hold true according to the order in the table, as far as the ear can judge with certainty. Those that are willing to try the experiment, may readily do it by the help of the third column of the table, shewing the musical intervals answering to the respective consonances. But the analogy will be plainer perceived by intermitting several consonances, and trying it, for example, in this series of all the concords not exceeding the octave; VIII, V, 4th, VI, III, 3^d, 6th; but then they should not be tempered as usual, but tuned perfect. And if the experimenter be skilful in melody and composition, he must endeavour, as much as possible, to divest himself of all habitual prepossessions in favour of this or that concord, or succession of concords, acquired from the rules and practice of his art; in order to an impartial judgment of the simple perception of the smoothness and sweetness of each concord, and a fair comparison of such perceptions only.

11. Though nature has appointed no certain limit between concords and discords, yet as musicians distinguish consonances by those names for

their own uses, I may do the like for mine ; calling unisons, III^{ds} , v^{ths} and vi^{ths} , and their complements to the viii^{th} and compounds with viii^{ths} , Concords, and all other consonances, Discords.

12. If the times of the single vibrations of any two sounds be V and v , and if $V : v :: R : r$, representing the least integers in that ratio ; the length of the cycle of times between the successive coincidences of the pulses of V and v , is $rV = Rv$. Because these multiples of V and v are the least of any that can be equal.

For the same reason, if $V : x :: S : s$ in the least integers, the cycle $sV = Sx$.

13. Hence the length of the cycle of V and v , is to that of V and x , as r to s ; that is, the cycles of consonances that have a common sound or vibration V , are proportional to the Numerators of the fractions $\frac{r}{R}V = v$, $\frac{s}{S}V = x$, expressing the times of the single vibrations of the other sounds, as in Fig. 3, or to the lesser terms of the ratios in the first column of the table of the order of the simplicity of consonances.

14. Consequently were the degrees of simplicity of consonances to be estimated by the frequency of the coincidences of their pulses, or the shortness of their cycles, as is commonly supposed ; the unisons, viii^{ths} , $\text{viii} + \text{v}^{\text{ths}}$, $2 \text{viii}^{\text{ths}}$, $2 \text{viii} + \text{iii}^{\text{ds}}$, &c, whose cycles are but 1 vibration of the base, would be equally simple ; and the same may be said of the v^{ths} , $\text{viii} + \text{iii}^{\text{ds}}$, $2 \text{viii} + \text{T}^{\text{s}}$, &c, whose several cycles are but 2 vibra-

vibrations of the base; and the same also of all consonances having the same number for the lesser term of their perfect ratios; which shews that the frequency of coincidences is, of itself, too general a character of the simplicity or smoothness of a consonance, and therefore an imperfect one.

SECTION IV.

Of the antient Systems of perfect consonances.

1. **I**F no other primes but 1, 2, 3 were admitted to the composition of perfect ratios, a system of sounds thence resulting could have no perfect thirds; nor any perfect consonance whose vibrations are in any ratio having the number 5, or any multiple of it, for either of its terms, as 5 to 4, 6 to 5, 10 to 9, 16 to 15, &c: it being impossible for any powers and products of the given primes 1, 2, 3 to compose any other prime or multiple of it.

2. Fig. 3. The minor tones *DE*, *GA* being thus excluded, and major tones being put in their places, every perfect major III^d will be increased by a comma, as being the difference of the tones (*t*); and every hemitone and perfect minor 3^d will be as much diminished; because the 4^{th}

B 4 and

(*t*) Sect. II. Art. 4.

and v^{ths} , as CF and Fc , cG and GC , are perfect, whether 5 be admitted or not, as depending on the primes 1, 2, 3, only.

3. These diminished hemitones being called Limmas, the octave is now divided into 5 major tones and 2 limmas; as represented to the eye in Plate II. Fig. 6; where the consonances whose vibrations are expressed by such high terms as the powers of 8 and 9, &c, must needs be disagreeable to the ear, according to the foregoing analogy between the agreeable smoothness of a consonance and the simplicity of the numbers expressing the ratio of its vibrations (u): and that in reality they are so, any one will soon find if he pleases to try the following experiment.

4. Fig. 6. Ascending by a perfect v^{th} and descending by a perfect 4^{th} alternately, upon an organ or harpsichord tune the following sounds, from F to C , C to G , G to D , D to A , A to E , E to B , and the octave Ff will then be divided into 5 major tones and 2 limmas; because the differences of those successive v^{ths} and 4^{ths} are major tones.

Then having tuned perfect octaves to every one of those notes, try the consonances that would be perfect if the number 5 were admitted, as thirds major and minor, with their complements to the $viii^{\text{th}}$ and compounds with $viii^{\text{ths}}$; and you will find them extremely disagreeable (x).

5. But

(u) Sect. III. Art. 10.

(x) The v^{ths} and 4^{ths} being tuned by the judgment of the ear, if any one doubts whether their single vibrations be

5. But if 5 be admitted among the musical primes, the ratios 10 to 9 and 16 to 15, belonging to the minor tone and the hemitone, are also admitted, and the elements that now compose the octave, are 3 major tones, 2 minor and 2 hemitones, as in Fig. 3.

PROPOSITION I.

A system of sounds whose elements or smallest intervals are tones major and minor and hemitones, will necessarily contain some imperfect concords.

6. Whatever be the order of those elements in any one octave, it must be the same in every one; to the end that every sound may have a perfect octave to it, as being the best concord. And in order to have as many perfect v^{ths} as possible, and consequently $viii + v^{\text{ths}}$, which concords are the second best (y), the elements must be ranged in such order, that the contiguous couples shall make as many perfect thirds as possible, both major

be as 3 to 2 and 3 to 4, let the Musician compare the sound of $\frac{2}{3}$ of a musical string, and also $\frac{3}{4}$ of it with that of the whole, and he will acknowledge these concords and those which he tuned upon the instrument to be the same, and of consequence to have the same ratios of their single vibrations.

(y) See the table of the order of concords in Sect. III. Art. 5.

major and minor; these being the intervals which compose the perfect v^{ths} . And that order being rightly determined, we shall have the greatest number of perfect concords of all sorts. Because the complements to the octave, of perfect thirds and v^{ths} , will also be perfect, and so will their compounds with any number of $viii^{\text{ths}}$.

Now it is observable of the seven elements T, T, T, t , t , H, H, which compose an octave, that T and H, T and t are the only couples which make perfect thirds (z), all the rest, T and T, t and t , t and H, H and H, making thirds imperfect by a comma, except H and H, which compose an imperfect tone, bigger than the major tone by almost a comma (a).

Hence either T and H, or T and t must be the outermost elements in the octave, as in the following table.

For if the first element in every octave in the system be T and the seventh be H, the seventh in any octave, combined with the first in the next octave, will compose the interval $H + T$ of a perfect

(z) Sect. II. Art. 5 and 7.

$$(a) \text{ Putting } H = \log. \frac{16}{15} = 0.02803$$

$$\text{Then } 2H = 2 \times \log. \frac{16}{15} = 0.05606$$

$$\text{And } T = \log. \frac{9}{8} = 0.05115$$

$$\text{Whence } 2H - T = \overline{0.00491}$$

$$\text{And the Comma} = \log. \frac{81}{80} = 0.00540$$

$$\text{Difference} \quad \overline{0.00049}$$

perfect minor 3^d , and thus the contiguous octaves will be joined in perfect concord.

Table of the Elements.

&c		7	1	2	3	4	5	6	7	1 &c
Cas. 1.	{	H	T + t		H + T + t			T + H		T
					t + T + H					
Cas. 2.	{	H	T + H		t + T + t			T + H		T
	{	t	T + t		H + T + H			T + t		T
					t + T + H					
	{	t	T + H					T + t		T
					H + T + t					

Likewise if the first element in every octave be T and the seventh be t, here also the seventh in any octave, together with the first in the next octave, will compose the interval $t + T$ of a perfect major III^d , and thus the contiguous octaves will again be joined in perfect concord; and in no other case besides those two, as appears by the observation above.

Cas. 1. Now if the second element be t, the first joined to it composes the perfect major III^d , $T + t$. And if the sixth element be T, the seventh joined to it will compose the perfect minor third $T + H$.

Two of the seven elements in the octave being thus disposed of at each end of it, the contiguous
couples

couples of the remaining three cannot compose perfect thirds in any order different from this, $H+T+t$, or its reverse $t+T+H$; both which being transferred into the interval between those extreme couples, shew, that the elements in the second and third places, compose either the imperfect minor third $t+H$, or the imperfect major third $t+t$.

If H be the second element, as in the third rank of the table, the first couple does now compose the perfect minor third $T+H$, and the last being also $T+H$, as before, the three remaining elements must have this order $t+T+t$, to make perfect thirds of their contiguous couples; and being thus transferred into the interval between those extreme couples, they shew, that the second and third elements do again compose an imperfect minor third $H+t$.

Cas. 2. Here also the sixth element must be T , since no other joined to the seventh can make a perfect third, as $T+t$.

Now if the second element be t , this joined to the first makes the perfect major third $T+t$. And two of the seven elements in the octave being thus joined at each end of it, the contiguous couples of the remaining three, cannot compose the intervals of perfect thirds in any order different from this, $H+T+H$; which being transferred into the interval between the extreme couples, shews, that the second and third elements do here also compose the interval $t+H$ of an imperfect minor third.

If

If H be the second element, as in the next lower ranks, then the first couple compose the interval $T+H$ of a perfect minor 3^d, and the last couple being $T+t$ as before, the three remaining elements must have this order, $t+T+H$, or its reverse, $H+T+t$, for the reason above; and being thus transferred into the middle interval, they shew, that the elements in the second and third places do again compose an imperfect minor third, $H+t$, or else an imperfect tone $H+H$; which being joined to the major tone on either side of it, composes an imperfect major third, greater than $t+T$ by almost two commas, as appears by the preliminary observation.

Now any one of those imperfect minor thirds, $t+H$, together with the contiguous perfect major III^d, composes a fifth equally imperfect, and so does the imperfect major third $t+t$ with the perfect minor third next to it. And the complements to the VIIIth of these imperfect concords, as well as their compounds with VIII^{ths}, are also equally imperfect, which proves the proposition. For having shewn the necessary defects in those six arrangements of the seven elements, we are freed from the trouble of considering the rest (b).
Q. E. D.

7. *Coroll.* Of those six arrangements of the elements, the first and fifth in the table are equally good

(b) Mr. *De Moivre's* general corollary to the xvi problem of his *Doctrine of Chances*, gives 210 permutations of these seven things, T, T, T, t, t, H, H.

good, and better than any one of the rest, as producing as many perfect thirds, and a greater number of perfect v^{ths}.

Pl. II. Fig. 7. In order to enumerate them with certainty and ease, if the circumference of a circle, be divided into seven arches, CD , DE , EF , FG , GA , AB , BC , proportional to T , t , H , T , t , T , H , placed in the respective angles at the center; they and their sums, whether smaller or greater than the circumference, here considered as a continued spiral, will represent all the intervals in a system composed of any number of octaves, and the corresponding intervals in different octaves will be denoted by the same arch and letters: as appears by conceiving the base of the third Figure coiled round into the circumference of a circle, equal to the line Cc or $c' \&c$. (c)

In this notation then we have only three major III^{ds}, CE , FA , GB , and they all perfect; and four minor thirds, DF , EG , AC , BD , the first of which being composed of $t+H$, instead of $T+H$,

(c) In this notation of intervals by circular arches, that the reader may not be at a loss for a suitable notation of the lengths of the corresponding homogeneous strings; let the radius OC be 1 and in OD , OE , OF , OG , OA , OB , OC , from the center set off $\frac{8}{9}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{8}{15}$, $\frac{1}{2}$ of the radius. These are the same lengths as those of the Monochord in Fig. 2, or Fig. 3; and as a regular curve drawn thro' the ends of the parallel strings in Fig. 3, is a Logistic Line whose Asymptote is the line Cc , so a regular curve drawn thro' the ends of the diverging strings in Fig. 7. is an Equiangular Spiral whose Pole is the center of the circle. See Sect. I. Art. 10. and Mr. Cotes's Harmonia Mensurarum, Prop. V and VI,

T+H, is too small by a comma; and six fifths, *FAC*, *CEG*, *GBD*, *DFA*, *ACE*, *EG B*, all perfect but *DFA*, which being composed of the defective minor third *DF* and the perfect major III^d *FA*, is too small by a comma.

These imperfections being caused by the contiguity of *t* and *H* in the cycle of the elements, cannot be avoided while the hemitones are separated; there being but 3 major tones in the cycle; and if they be joined, as in Fig. 12, the consequences will be worse.

The rest will appear by enumerating the thirds and fifths in the 8th, 9th, 10th, 11th, and 12th Figures, made according to the other five arrangements in the Table of Elements (*d*).

8. Now if any one pleases to try the following experiment, he will find what effect these imperfect fifths and fourths and their compounds with VIII^{ths}, will have upon the ear; that of the thirds and sixths having been tried before (*e*).

In

(*d*) Sir *Isaac Newton* happily discovered, (*Optics* Book 1, Part 2, Prop. 3) that the breadths of the seven primary colours in the sun's image, produced by the refraction of his rays through a prism, are proportional to the seven differences of the lengths of the eight musical strings, *D, E, F, G, A, B, C, d*, when the intervals of their sounds are *T, H, t, T, t, H, T*: which order is remarkably regular; but though it agrees best with the prismatic colours, it is not the properest for a system of concords, as producing one major third, two minor thirds and two fifths severally imperfect by a comma. See Fig. 13. N^o. 2.

(*e*) Sect. iv. Art. 4.

In Fig. 3, tune upwards from *C* the two perfect v^{th} *CG*, *Gd*, and the perfect $xvii^{\text{th}}$, or $2viii + iii$, *Cé*, then downwards the v^{th} *éa*, and the intermediate fifth *ad* will be too little by a comma, as including the imperfect minor third *df*. And by tuning an eighth below *a* we have the imperfect fourth *Ad* too large by a comma.

9. The disagreeable effect of this fifth *da* and fourth *dA* in every octave, and of their compounds with $viii^{\text{th}}$, and also of the third *df* and and sixth *fd'* in every octave and of their compounds with $viii^{\text{th}}$, and of many more such imperfect concords, when the usual flat and sharp sounds are added to complete the scale, has obliged practical musicians, long ago, to distribute that comma, wanting in the fifth *da*, equally among all the four v^{th} , *CG*, *Gd*, *da*, *aé*, contained in the $xvii^{\text{th}}$ *Cé*. And this interval *Cé* may be increased or decreased a little before it be divided into 4 equal v^{th} . In any case such distribution is therefore called the Participation or Temperament of the system, and when rightly adjusted is undoubtedly the finest improvement in harmonics.

10. If it be asked why no more primes than 1, 2, 3, 5 are admitted into musical ratios; one reason is, that consonances whose vibrations are in ratios whose terms involve 7, 11, 13, &c, *cæteris paribus* would be less simple and harmonious (*f*)
than

(*f*) Sect. III. and Table of the order of the simplicity of consonances.

than those whose ratios involve the lesser primes only.

Another reason is this; as perfect fifths and other intervals resulting from the number 3, make the Schism of a comma with the perfect thirds and other intervals resulting from the number 5, so such intervals as result from 7, 11, 13, &c, would make other schisms with both those kinds of intervals.

11. The Greek musicians, after dividing an octave into two 4^{ths}, with the diazeuctic or major tone in the middle between them, and admitting many primes to the composition of musical ratios, subdivided the 4th into three intervals of various magnitudes, placed in various orders, by which they distinguished their Kinds of Tetrachords (g). Two of them have occurred in this Section. The first, or $\frac{3}{4} = \frac{2}{3} \times \frac{3}{2} \times \frac{2}{3}$, answering to the 4th = T + T + L, in Fig. 6, is *Ptolemy's Genus Diatonum ditonicum*, and results from that division of a Monochord which bears the name of *Euclid's Section* of the Canon; the second Kind, or $\frac{3}{4} = \frac{2}{3} \times \frac{2}{5} \times \frac{5}{4}$, answering to the 4th = T + t + H, in Fig. 3, is *Ptolemy's Diatonum intensum*.

12. Since the invention of a temperament, all those antient systems have justly been laid aside, as being unfit for the execution of musi-

(g) Dr. Wallis has given a table of them in his Appendix to *Ptolemy's Harmonics*. Oper. Math. vol. III, pag. 166.

cal compositions in several parts. But to conclude from thence that the antients had no music in parts, would be a very weak inference. Because it is much easier for practical musicians to follow the judgment of the ear, which leads naturally to an occasional temperament of any disagreeable concords, than to learn and put in practice the theories of philosophers (*b*): And also

(*b*) It may not be amiss to add the opinion of the famous *Salinas*. Sed unum hoc omnes scire volo, instrumenta quibus antiqui utebantur, consonantias habuisse imperfectas, ut ea, quibus nunc utimur. Neque enim aliter modulatio convenienter exerceri poterat. Quod si de hac consonantiarum imperfectione, neque *Ptolemæus*, neque alius ex antiquis musicis mentionem fecisse reperitur, causam potissimam esse crediderim, quòd ad practicos eam pertinere arbitrarentur; quoniam sensu duce solùm, non arte aut ratione semper fieri solita sit: cujus plenissimum et evidentissimum testimonium reperitur apud *Galenum*, libro primo De Sanitate tuenda, capite quinto; ubi magnam esse latitudinem sanitatis ostendere volens, sic inquit: Καὶ τί θαυμασὸν εἰ τὴν εὐκρασίαν εἰς ἱκανὸν ἐκλείναισι πλάττονται πάντες, ὅπερ γὰρ καὶ ἐν αὐταῖς λύραις εὐαρμοσίαν, τὴν μὲν ἀκριβεστάτην δῆπερ, μίαν καὶ ἀτμητοὺν ὑπάρχουσαν εἰκός· ἢ μὲν τοι γὰρ εἰς χρεῖαν ἴστα, πλάττονται ἔχει. Πολλὰ γὰρ ἐν ἡρμόδῳ δοκῶσαν ἄριστα λύραν, ἕτερον μουσικὸς ἀκριβῶς ἐφημερόσατο· πανταχῶς γὰρ ἡ αἰδήσις ἡμῶν ὅτι χρητήριον, ὡς πρὸς τὰς ἐν τῇ βίῳ χρεῖας. hoc est, Quid mirum, si *Eucrasiam* in satis amplam latitudinem extendunt universi; quando et in lyris consonantiam ipsam quæ summa exactissimaque sit, unicam atque insectabilem esse probabile sit, et quæ in usus hominum venit, certe latitudinem habeat. Sæpe namque, [quam] percommode temperasse lyram videaris, alter superveniens musicus exactius temperavit: siquidem nobis ad omnia vitæ munera sensus ubique judex est. Ex quibus *Galen*i verbis liquido constat, consonantias, quibus

also because we are assured from history, that experience and necessity did introduce something of a temperament before the reason of it was discovered, and the method and measure of it was reduced to a regular theory, as in the following proposition.

SECTION V.

Of the temperaments of imperfect intervals and their synchronous variations.

PROPOSITION II.

To reduce the diatonic system of perfect consonances to a tempered system of Mean Tones.

PLATE III. Fig. 13. When the elements are ranged in this order, T, t, H, T, t, T, H, or this, t, T, H, T, t, T, H, which two we shewed to be the best (i), and the arches CD ,
C 2
 DE ,

bus in musicis utebantur instrumentis, jam tunc imperfectas esse, quin potius et fuisse semper et semper esse futuras. De Musicâ lib. III. cap. 14. Be it so; but did they know, that all the concords cannot be tuned perfect, and why they cannot?

(i) Sect. IV. Art. 7.

DE , EF , FG , GA , AB , BC , are proportional to them, let the major 111^d CE , situated between the two hemitones, be bisected in d ; and let the other two major tones, FG , AB , be diminished at both ends by the intervals Ff , Gg , Aa , Bb , severally equal to half Dd ; and the octave will then be divided into five mean tones and two limmas, each limma being bigger than the hemitone by a quarter of a comma.

For the interval Dd being half the difference between the major and minor tones, CD , DE , is half a comma (k), and therefore the new tone Cd or dE is an arithmetical mean between them. And each of the temperaments Ff , Gg , Aa , Bb , being made equal to half Dd or a quarter of a comma, it appears that every major tone is diminished by half a comma, and that every minor tone is as much increased, which reduces all the tones to an equality. And by the construction the limmas bC , Ef exceed the hemitones by a quarter of a comma apiece.
Q.E.D.

Coroll. In the system of mean tones every perfect v^{th} is diminished by a quarter of a comma: as will appear by going round the 13^{th} figure, and comparing the tempered v^{ths} , faC , CEg , gbd , dfa , aCE , Egb , with the perfect ones, by means of the notes T , t , H in the angles.

This

This is usually called the vulgar temperament and might be proved several other ways independent of the first and second propositions (1).

PRO-

(1) *Salinas* tells us, that when he was at Rome, he found the musicians used a temperament there, though they understood not the reason and true measure of it, till he first discovered it, and *Zarlino* published it soon after; first in his *Dimonstrationi Harmoniche*, *Ragionamento quinto*, *proposta 1^{ma}*, and after that, in his *Institutioni Harmoniche*, part. 2. cap. 43.

After his return into Spain, *Salinas* applied himself to the latin and greek languages, and caused all the antient musicians to be read to him, for he was blind; and in 1577 he published his learned work upon music of all sorts; where treating of three different temperaments of a system, he prefers the diminution of the vth by a quarter of a comma to the other two, which he says are peculiar to certain instruments. *De Musicâ Lib. III. cap. 22.*

Dechales says, that *Guido Aretinus* was the inventor of that temperament: Ipse nulla habita ratione toni majoris et minoris, hunc unius quintæ defectum aliis omnibus quintis communicat, et quasi dividit, ita ut nulla deficiat nisi quarta parte commatis. Hoc systema, quod valde commodum est, dicitur *Aretini*. *Cursus Mathem. Tom. I. pag. 62. De Progressu matheſeos et musicæ, cap. 7; et Tom. IV. pag. 15. cap. XI.* But that opinion wants confirmation, especially as *Dechales* makes no mention of the claims of *Zarlino* and *Salinas* to that invention; for it seems they had a dispute about it.

PROPOSITION III.

If the five mean tones and the two limmas, that compose a perfect octave, be changed into five other equal tones and two equal limmas, of any indeterminate magnitudes; the synchronous Variations of the limma L, the mean tone M, and of every interval composed of any numbers of them, are all exhibited in the following table, by the numbers and signs of any small indeterminate interval v : And are the same quantities as the variations of the temperaments of the respective imperfect intervals.

For

2^d	II^d	3^d	III^d	4^{th}	IV^{th}
L	M	$L+M$	$2M$	$L+2M$	$3M$
$5v$	$-2v$	$3v$	$-4v$	v	$-6v$
$-5v$	$2v$	$-3v$	$4v$	$-v$	$6v$
$L+5M$	$2L+4M$	$L+4M$	$2L+3M$	$L+3M$	$2L+2M$
VII^{th}	7^{th}	VI^{th}	6^{th}	V^{th}	5^{th}

For since the perfect $\text{VIII}^{\text{th}} = 2L + 5M$ is invariable, if the variation of L be put equal to $5v$, as in the table, that of $2L$ is $10v$, and that of $5M$, as being the complement of $2L$ to the VIII^{th} , is $-10v$; whence the variation of M is $-2v$.

Consequently the variation of the mean 3^{d} , $L + M$, is $5v - 2v = 3v$, and that of the III^{d} , $2M$, is $-4v$, and that of the mean 4^{th} , $L + 2M$, is $5v - 4v = v$, and that of the mean IV^{th} , $3M$, is $-6v$.

The variations of the intervals in the lower half of the table, are respectively equal to those in the upper half, but have contrary signs; the corresponding intervals being complements to the perfect octave.

For which reason the compounds of every one of those intervals with any number of octaves, have respectively the same variations both in quantity and quality.

And if the sign of the variation of any one interval be changed, the signs of all the rest will also be changed; because their quantities will vanish all together when v or any one multiple of it vanishes.

As to the second part of the proposition, it will appear in Fig. 13, that any variation v of the mean interval $CdEf$ is the same in quantity as the variation of the temperament Ff of the said interval $CdEf$: and the like is evident in any other instance. Q. E. D.

Coroll. 1. It is observable in the table, that the variations of all the major mean intervals II^d , III^d , IV^{th} , v^{th} , VI^{th} , VII^{th} , have the same sign, and those of the minor intervals the contrary sign.

Coroll. 2. Having extended the circumference $CdEfgabC$ of Fig. 13 into a right line, as in Fig. 14, at the points d, E, g, a, b , that terminate the major mean intervals II^d , III^d , v^{th} , VI^{th} , VII^{th} , measured from C , (and the minor too measured from c the other extreme of the octave Cc) place the respective tabular numbers 2, 4, 1, 3, 5, denoting the proportions of their synchronous variations; and in Fig. 15 divide any given line $O6$ into 6 equal parts, at the points 1, 2, 3, 4, 5; then conceive the 14th Fig. transferred to the 15th five several times, into five parallel positions, so that the several points 1, 2, 3, 4, 5 in each Figure may coincide. And it will be evident, by coroll. 1, that any right line $Ovvvv$, drawn from O , terminates the synchronous variations, $1v$, $2v$, $3v$, $4v$, $5v$, of those mean intervals, v^{th} , II^d , VI^{th} , III^d , VII^{th} , the variations being measured from their respective origins 1, 2, 3, 4, 5; and that these are also the synchronous variations of the temperaments of the respective imperfect intervals, and of their complements to the $VIII^{th}$ and compounds with $VIII^{th}$ s, that is, of all the intervals in the system.

For as to the mean IV^{th} fb , Fig. 14, its contemporary variation in Fig. 15, will be the line

6v

6v in the sixth parallel $F6B'F'$, when its temperament $B'6$ or $B'b'$ is taken equal to twice Bb and placed the same way from its origin b' or 6. Because in Fig. 14 the temperament of the imperfect iv^{th} fb is $Ff + Bb = 2Bb$.

As want of room in Fig. 15 will not permit the several intervals CG , CD , &c. even less than one octave, to be represented in their due proportions to $G1$, the quarter of the comma, which is but the 223^d part of an octave; we must conceive them continued far beyond the margin of the paper.

Coroll. 3. When the III^d is perfect, the temperaments belonging to the v^{th} and vi^{th} are severally $\frac{1}{4}$ of a comma, the former in defect, the latter in excess: and if either of them be made less, the other will be greater than $\frac{1}{4}$ comma.

Pl. III. & IV. Fig. 15, 16. For when the Temperer $Ovvv$ falls upon E , the III^d CE is perfect, and the tempered v^{th} $C1$ is less than the perfect v^{th} CG by $G1$, and the tempered vi^{th} $C3$ is bigger than the perfect vi^{th} CA by $A3 = G1 = \frac{1}{4}$ comma.

Hence when Av , any other temperament belonging to the vi^{th} , is less than $A3$ or $\frac{1}{4}$ comma, Gv the corresponding temperament belonging to the v^{th} , is greater than $G1$ or $\frac{1}{4}$ comma: and on the contrary, when Gv is less than $G1$, the respective Av is bigger than $A3$. And whatever be the magnitudes of these temperaments of the v^{th} and vi^{th} , those of their com-

complements to the $viii^{th}$ and compounds with $viii^{th}$ s are the same.

Coroll. 4. When the vi^{th} is perfect, the temperaments of the v^{th} and iii^d are severally $\frac{1}{4}$ comma, and are both negative.

Fig. 16. For when the temperer $Ovvv$ falls upon the line $OHAI$, the temperament of the vi^{th} vanishes, and those of the v^{th} and iii^d are GH and EI , and are equal. For the equal lines GI , A_3 and equal triangles GEO , AOE shew, that the line GE is parallel to AO ; whence GH is equal to EI , and the similar triangles IEO , A_3O give $IE = \frac{4}{3}A_3 = \frac{4}{3} \times \frac{1}{4} \text{comma} = \frac{1}{3} \text{comma}$.

Coroll. 5. When the v^{th} is perfect, the temperaments of the vi^{th} and iii^d are severally equal to a comma in excess.

For when the temperer $Ovvv$ falls upon the line $OGKL$, the temperament of the v^{th} vanishes, and those of the vi^{th} and iii^d are now AK and EL , which are equal, because of the parallelograms $AEGO$, $AELK$; and EL is $= 4 \times GI$ or four quarters of a comma.

Coroll. 6. When the temperer $Ovvv$ falls within the angle AOE , the $\text{temp}^t. v^{th} = \text{temp}^t. vi^{th} + \text{temp}^t. iii^d$, that is, the line $Gv = Av + Ev$, or the lines $GI + Iv = A_3 - 3v + Ev$, that is, putting the letter v for the line Iv , $\frac{1}{4}c + v = \frac{1}{4}c - 3v + 4v$, which is evidently true.

Coroll. 7. When the temperer $Ovvv$ falls within the angle EOG , the $\text{temp}^t. vi^{th} = \text{temp}^t. v^{th} + \text{temp}^t. iii^d$, that is, the line $Av = Gv + Ev$,

Ev , or the line $sA_3 + 3v = GI - Iv + Ev$, that is, putting v for the line Iv , $\frac{1}{4}c + 3v = \frac{1}{4}c - v + 4v$, which is true.

Coroll. 8. When the temperer falls any where out of the angle AOG , the temp^t. $III^d = \text{temp}^t. v^{\text{th}} + \text{temp}^t. vI^{\text{th}}$, that is, when it falls beyond the side AO , the temp^t. $EI + Iv = GH + Hv + Av$, or putting the letter v for the line Hv , $\frac{1}{3}c + 4v = \frac{1}{3}c + v + 3v$, which is true: and when the temperer falls beyond the other side OG , the said temp^t. $EL + Lv = Gv + AK + Kv$, that is, putting v for the line Gv , $c + 4v = v + c + 3v$, which is true.

Coroll. 9. The sum of the temperaments of the v^{th} and vI^{th} is $\frac{1}{2}$ a comma when the III^d is perfect; is less than $\frac{1}{2}$ a comma by $\frac{1}{2}$ the temperament of the III^d when flattened; and greater than $\frac{1}{2}$ a comma by $\frac{1}{2}$ the temperament of the III^d when sharpened.

For in the first case the said sum is $GI + A_3$; in the second, it is $GI + Iv + A_3 - 3v = GI + A_3 - 2v$; and in the third, it is $GI - Iv + A_3 + 3v = GI + A_3 + 2v$; in which latter cases the temperament of the III^d is $4v$.

Coroll. 10. Hence the sum of the temperaments of all the concords is less when the III^{ds} are flattened, than the like sum when the III^{ds} are equally sharpened; and the sum is the least of all when the III^{ds} are perfect, as in the system of mean tones (m).

(m) Prop. II.

Scholium.

Scholium.

From the third and tenth corollaries I think we might justly pronounce the system of mean tones to be the best possible, were it evident that equal temperaments cause different concords to be equally disagreeable to the ear (*n*).

But if it shall appear, that the v^{th} and 3^d and their compounds with octaves, are more disagreeable in their kind, than the v^{th} and 4^{th} and their compounds with octaves, all being equally tempered, as in that system; will it not follow, that the temperament of the former Parcel of concords should be smaller than that of the latter, to make them all as equally harmonious as possible, without spoiling the harmony of the III^d and 6^{th} and their compounds with octaves; which third parcel makes up the sum of all the concords in the system.

For

(*n*) Mr. *Huygens* has pronounced it the best, in saying that the musicians in the other planets may know perhaps, *cur optimum sit temperamentum in chordarum systemate, cum ex diapente quarta pars commatis ubique deciditur*; *Cosmotheoros* pag. 76; but has given us no reason for his assertion, either in that incomparable book or in his *Harmonic Cycle*; where he only appeals to the approbation and practice of musicians and refers to the demonstrations of *Zarlino* and *Salinas*. But neither of these celebrated authors do any thing more, if I rightly remember, (for I have not the books now by me) than reduce the Diatonic system of perfect consonances to that of mean tones, by distributing the schism of a whole comma into quarters; not at all considering, whether those equal temperaments have the same, or a different effect upon the several concords.

For if it be the immediate succession of a worse harmony to a better, as in instruments badly tuned, which chiefly offends the ear ; it must be allowed, that a system would be the better, *cæteris paribus*, for having all the concords as equally harmonious in their kinds, as the nature and properties of numbers will permit.

In order to resolve those questions upon philosophical principles, and to determine the temperament of a given system, that shall cause all the concords, at a medium of one with another, to be equally, and the most harmonious in their several kinds, I found it necessary to make a thorough search into the abstract nature and properties of tempered consonances ; and thence to derive their effects upon our organs of hearing : A large field of harmonics hitherto uncultivated.

But before I enter upon it, it will be convenient to finish this section with a determination of the least sum of any three temperaments in different parcels, when any two of them have any given ratio.

P R O-

PROPOSITION IV.

To find a set of temperaments of the v^{th} , v_1^{th} and III^{d} upon these conditions; that those of the v^{th} and v_1^{th} shall have the given ratio of r to s , and the sum of all three shall be the least possible.

Pl. V. VI. Part of the 17^{th} and 18^{th} Figures being constructed like the 15^{th} , from A towards K take $AM:G_1::s:r$, and through the intersection p of the lines AG , M_1 , draw the temperer $Orst$; I say Gr , As , Et are the temperaments required.

For by the similar triangles Grp , Asp , and G_1p , AMP , we have $Gr:As::(Gp:Ap::G_1:AM::) r:s$ by construction, as required by the first condition.

Again, in the same line MAC take $AN=AM$, and through the intersection P of the lines AG , N_1 produced, draw another temperer $ORST$; and by the similar triangles GRP , ASP , and G_1P , ANP , we have $GR:AS::(GP:AP::G_1:AN \text{ or } AM::) r:s$ by construction, which likewise answers the first condition; and it is easy to understand, that no other temperers but those two can answer that condition.

Now

Now whatever be the quantity and quality of the given ratio r to s , I say the sum $Gr + As + Et$ is less than $GR + AS + ET$.

Case 1. Fig. 17. For when r is bigger than s , or the ratio of r to s , or of $G1$ or $A3$ to AM or AN , is a ratio of majority, the temperers Op , OP fall within the angles AOE , AOC respectively; as appears by the construction. Whence, by coroll. 6 and 8. prop. III, $Gr = As + Et$, and $ET = GR + AS$; and therefore $Gr + As + Et : GR + AS + ET :: Gr : ET$, which is a ratio of minority, because Gr is less than GH or $EI(o)$ and EI less than ET .

Case 2. Fig. 18. When r is less than s , or the ratio of r to s , or of $G1$ to AM or AN is a ratio of minority, the temperers Op , OP fall within the angles EOG , AOC respectively; as appears by the construction. Whence, by coroll. 7 and 8. prop. III, $As = Gr + Et$, and $ET = GR + AS$, and therefore $Gr + AS + Et : GR + AS + ET :: As : ET$, which is a ratio of minority; because $Gr : As :: r : s :: GR : AS$, whence, as Gr is less than GR , so As is less than AS , which is less than IT , which is less than ET .

Case 3. Fig. 17 and 18. When r to s , or $G1$ to AM or AN , is the ratio of equality, the temperer $Orst$ coincides with the line OE , and $Orst$ is parallel to GA ; whence it is plain, that the sum of the temperaments $G1 + A3 + o$, is less than $GR + AS + ET$, as required. Q. E. D.

Coroll.

(o) See coroll. 4. Prop. III.

Coroll. Putting c for the comma EL or four $G1$, when the temp^t. $v : \text{temp}^t. vi :: r : s$, the required temperaments of the v , vi and iii are, $Gr = \frac{r}{3r+s} c$, $As = \frac{s}{3r+s} c$ and $\pm Et = \frac{r-s}{3r+s} c$. And according as r is bigger or less than s , the temperer $Orst$ falls within the angle AOE or EOG .

Fig. 17 and 18. For, $As : Gr :: s : r$, and $As : 3Gr$ or $sK :: s : 3r$, and $As : As + sK$ or $c (p) :: s : s + 3r$. Whence $As = \frac{s}{3r+s} c$, and $Gr = \frac{r}{s} As = \frac{r}{3r+s} c$, and in the angle AOE , $Et = Gr - As = \frac{r-s}{3r+s} c$, but in EOG , $Et = As - Gr$, by the equations in case 1, 2.

PROPOSITION V.

To find a set of temperaments of the v^{th} , vi^{th} and iii^{d} upon these conditions; that those of the v^{th} and iii^{d} shall have the given ratio of r to t , and the sum of all three shall be the least possible.

Pl. VII. VIII. Fig. 19, 20. If t to r be a ratio of minority, or of equality, or even of majority less than 1 to $\frac{1+\sqrt{33}}{8}$ or 0.843070 &c, from E towards I take $EM : G1 :: t : r$, and through

(p) See Dem. coroll. 5. prop. III.

through the intersection p of the lines M_1, GE produced, draw the temperer $Orst$, and the required temperaments will be Gr, As, Et .

But if the ratio of t to r be greater than 1 to 0.843070 &c, in Fig. 20, from E towards L take $EN : G_1 :: t : r$, and through the intersection P of the lines N_1, GE , draw the temperer $ORST$, and the required temperaments will be GR, AS, ET .

And if $t : r :: 1 : 0.843070$ &c, the required temperaments will be Gr, As, Et , or GR, AS, ET , their sums being equal.

In the first case, Fig. 19, take $EN = EM$, and in the second, Fig. 20, $EM = EN$; and through the intersections P, p of the lines N_1, M_1 with GE , draw two more temperers $ORST, Orst$.

Then by the similar triangles $Grp, Et p$ and $G_1 p, EM p$, we have $Gr : Et :: (Gp : Ep :: G_1 : EM ::) r : t$ by construction, as required by the first condition.

Again, by the similar triangles GRP, ETP and $G_1 P, ENP$, we have $GR : ET :: (GP : EP :: G_1 : EN ::) r : t$ by construction, which also answers the first condition; and it is easy to understand that no other temperers but those can answer that condition.

Case 1. Fig. 19. Now when t is to r , and therefore EM or EN to G_1 in a ratio of minority, the temperers Op, OP fall within the angles AOE, EOG respectively by the construction.

D

struction. Whence, by coroll. 6 and 7. prop. 3,
 $Gr = As + Et$ and $AS = GR + ET$.

But $Gr : Et :: r : t$, and $Gr : \frac{1}{4} Et$ or rI
 $:: r : \frac{1}{4} t$, and $Gr : Gr - rI$ or $\frac{1}{4} c :: r : r -$
 $\frac{1}{4} t :: 4r : 4r - t$. Whence $Gr = \frac{r}{4r-t} c$, and
 $Et = \frac{t}{r} Gr = \frac{t}{4r-t} c$, and $As = Gr - Et =$
 $\frac{r-t}{4r-t} c$, by the equation in the last paragraph.

Likewise $Gr : ET :: r : t$, and $GR : \frac{1}{4} ET$ or
 $R I :: r : \frac{1}{4} t$, and $GR : GR + RI$ or $\frac{1}{4} c :: r :$
 $r + \frac{1}{4} t :: 4r : 4r + t$. Whence $GR = \frac{r}{4r+t} c$,
 and $Et = \frac{t}{r} GR = \frac{t}{4r+t} c$, and $AS = GR +$
 $ET = \frac{r+t}{4r+t} c$, by the equation above.

Therefore $Gr + As + Et : GR + AS + ET ::$
 $Gr : AS :: \frac{r}{4r-t} : \frac{r+t}{4r+t} :: 4rr + rt : 4rr +$
 $rt, + 2rt - tt$, which is a ratio of minority;
 because t being less than r , tt is less than $2rt$.

Case 2. Fig. 20. When t to r , and therefore
 EM or EN to GI , is a ratio of majority, the
 temperers Opt , OPT fall within the angles,
 AOC , EOG respectively; as appears by the
 construction. Whence, by coroll. 8 and 7 prop.
 3, $Et = Gr + As$ and $AS = GR + ET$.

In which case the theorems for the values of
 Gr , As , Et , GR , AS , ET are the same as
 before.

Therefore $Gr + As + Et : GR + AS + ET ::$
 $Et : AS :: \frac{t}{4r-t} : \frac{r+t}{4r+t} :: 4rt + tt : (4rr +$
 $4rt$

$4rt - rt - tt$ or) $4rt + tt$, $+4rr - rt - 2tt$, which is a ratio of minority, except either when $4rr - rt - 2tt = 0$, or $\frac{4rr - rt - 2tt}{4tt} = 0$, or $\frac{rr}{tt} - \frac{r}{4t} - \frac{1}{2} = 0$, which gives $\frac{r}{t} = \frac{1 + \sqrt{33}}{8} = 0.843070$ &c (q); or when $4rr - rt - 2tt$, or $\frac{rr}{tt} - \frac{r}{4t} - \frac{1}{2}$ is negative, and consequently $\frac{r}{t}$ is less than 0.843070 &c (r), or the ratio of t to r is greater than 1 to 0.843070 &c.

In the first case either $Gr + As + Et$ or $GR + AS + ET$, as being equal, are the required temperaments; in the second the latter only, as being less than the former.

Case 3. When $t = r$, we have EM or $EN = GI$; therefore the intersection p is removed to an infinite distance, and the temperer $Orst$ coincides

(q) For supposing $\frac{r}{t} = \frac{x}{1} = x$, we have $\frac{rr}{tt} - \frac{r}{4t} - \frac{1}{2} = xx - \frac{1}{4}x - \frac{1}{2} = 0$. Whence $xx - \frac{1}{4}x = \frac{1}{2}$ and $xx - \frac{1}{4}x + \frac{1}{8} \times \frac{1}{8} = \frac{1}{8} \times \frac{1}{8} + \frac{1}{2} = \frac{1}{8} \times \frac{1}{8} + \frac{32}{8 \times 8} = \frac{33}{8 \times 8}$ whose square roots are $x - \frac{1}{8} = \pm \frac{\sqrt{33}}{8}$; whence x or $\frac{r}{t} = \frac{1 \pm \sqrt{33}}{8} = \frac{1 \pm 5.744562}{8}$ &c.

(r) For since the root 0.843070 &c, when substituted for x , will make the value of $xx - \frac{1}{4}x - \frac{1}{2}$, or of $x - \frac{1}{4} - \frac{1}{2x} = 0$; a smaller number substituted for x , will produce a negative value of the latter, and consequently of the former quantity.

cides with $OHAI$. Hence $Gr + As + Et$ becomes $= GH + 0 + EI$, and is to $GR + AS + ET :: 5 : 6$, a ratio of minority, produced by putting $t=r$ in the terms of that ratio in case 1 or 2. Q. E. D.

Coroll. When the temp^t. $v : \text{temp}^t. III :: r :$
 t , if $\frac{r}{t}$ be bigger than 0.843070 &c, the required temperaments of the v , VI and III are,
 $Gr = \frac{r}{4r-t} c$, $As = \frac{r-t}{4r-t} c$, $Et = \frac{t}{4r-t} c$. And the temperer $Orst$ falls within the angle AOE or AOC , according as r is bigger or less than t .

But if $\frac{r}{t}$ be less than 0.843070 &c, they are
 $GR = \frac{r}{4r+t} c$, $AS = \frac{r+t}{4r+t} c$, $ET = \frac{-t}{4r+t}$; and the temperer $ORST$ falls within the angle EOG .

And if $\frac{r}{t} = 0.843070$ &c, their sums are equal and either of them answers the problem.

P R O-

PROPOSITION VI.

To find a set of temperaments of the v^{th} , vi^{th} and iii^{d} upon these conditions that those of the vi^{th} and iii^{d} shall have the given ratio of s to t , and the sum of all three shall be the least possible.

Pl. IX. X. Fig. 21 and 22. From E towards C take $EM : A_3 :: t : s$ and through the intersection p of the lines M_3 , AE draw the temperer $Orst$, and the required temperaments will be Gr , As , Et .

For by the similar triangles Asp , $Et p$ and $A_3 p$, $EM p$, we have $As : Et :: (Ap : Ep :: A_3 : EM ::) s : t$ by construction, as required by the first condition.

Again, taking $EN = EM$, through the intersection P of the lines N_3 , AE produced, draw the temperer $ORST$, and by the similar triangles ASP , ETP and $A_3 P$, ENP , we have $AS : ET :: (AP : EP :: A_3 : EN \text{ or } EM) :: s : t$ by construction, which also answers the first condition; and it is plain that those are all the temperers which can answer it.

Now whatever be the ratio of s to t , I say that $Gr + As + Et$ is less than $GR + AS + ET$.

Case 1. Fig. 21. When t is to s , or EM to A_3 in a ratio of minority, the temperers Op , OP fall within the angles AOE , EOG respectively, as appears by the construction. Whence by coroll. 6 and 7 prop. III, $Gr = As + Et$ and $AS = GR + ET$.

But $Et : As :: t : s$, and $Et : \frac{4}{3} As$ or $It :: t : \frac{4}{3} s$ and $Et : Et + It$ or $\frac{1}{3} c (s) :: t : t + \frac{4}{3} s :: 3t : 3t + 4s$. Whence $ET = \frac{t}{4s+3t} c$.

And $As = Et \times \frac{s}{t} = \frac{s}{4s+3t} c$. And $Gr = As + Et = \frac{s+t}{4s+3t} c$, by the equation in the last paragraph.

Again, $ET : AS :: t : s$ and $ET : \frac{4}{3} AS$ or $IT :: t : \frac{4}{3} s$ and $ET : IT - ET$, or IE or $\frac{1}{3} c :: t : \frac{4}{3} s - t :: 3t : 4s - 3t$. Whence $ET = \frac{t}{4s-3t} c$ and $AS = ET \times \frac{s}{t} = \frac{s}{4s-3t} c$.

Therefore $Gr + As + Et : GR + AS + ET :: Gr : AS :: \frac{s+t}{4s+3t} : \frac{4s-3t}{s} :: 4ss + 4st - 3st - 3tt$, or $4ss + 3st - 2st - 3tt : 4ss + 3st$, which is evidently a ratio of minority.

Case 2. Fig. 22. When t is to s , or EM to A_3 in a ratio of majority, the temperers Op , OP fall within the angles AOE , GOc respectively. Whence, by coroll. 6 and 8 prop. III, $Gr = As + Et$ and $ET = GR + AS$, and $Gr + As + Et : GR + AS + ET :: Gr : ET$, which is plainly a ratio of minority.

Case

(s) See Dem. coroll. 4. prop. III.

Case 3. When $t=s$ or EM or $EN=A3$, the intersection P vanishes, and the temperer $ORST$ coincides with $OGKL$, as appears by the construction. Whence by the conclusion of the second case, $Gr+As+Et : o+AK+EL :: Gr : EL$, a ratio of minority, as before. Q. E. D.

Coroll. When the temp^t. VI : temp^t. III :: $s : t$; the required temperaments of the v, VI and III are, $Gr = \frac{s+t}{4s+3t}c$, $As = \frac{s}{4s+3t}c$, $Et = \frac{t}{4s+3t}c$; and the temperer lies within the angle AOE , whatever be the quantity and quality of the ratio of s to t .

Scholium.

These three problems comprehend the solution of a more general one, namely, To find the temperament of a system of sounds upon these conditions; that the octaves be perfect, that the ratio of the temperaments of any two given concords in different parcels be given, and that the sum of the temperaments of all the concords, be the least possible.

The reason is, that the given ratio of the temperaments of any two concords, determines the position of the temperer of the system, and this the three magnitudes of the temperaments of all the concords, whatever be their number. But if both the given concords be contained in any one of the three parcels above men-

tioned (*t*), the given ratio of their temperaments can be no other than that of equality; and this *datum* is plainly insufficient.

SECTION VI.

Of the Periods, Beats and Harmony of imperfect consonances.

DEFINITIONS.

I. Any two sounds whose single vibrations have any small given ratio, are called Imperfect Unisons:

II. And the cycle of their pulses is called Simple or Complex, according as the difference of the least terms of that ratio is an unit or units:

III. And when a complex cycle is divided into as many equal parts as that difference contains units, each part is called a Period of the pulses:

IV. And the cycles of perfect consonances are often called Short cycles, to distinguish them from the long cycles of imperfect unisons.

P R O-

(*t*) Schol. prop. III.

PROPOSITION VII.

In going from either end to the middle of any simple cycle or period of the pulses of imperfect unisons, the Alternate Lesser Intervals between the successive pulses increase uniformly, and are proportional to their distances from that end; and at any distances from it less than half the simple cycle or period, are less than half the lesser of the two vibrations of the imperfect unisons.

Let the vibrations be V and v , and $V : v :: R : r$, the integers R, r being the least in that ratio; and putting $d = R - r$, we have the complex cycle $rV = Rv = rv + dv$ (u), and the period $\frac{r}{d} V = \frac{r}{d} v + v$, which when $d = 1$, is a simple cycle (x).

Pl. XI. Fig. 23, 24, 25, 26. To assist the imagination, let the successive vibrations V, V, V , &c, be represented by the equal lines AB, BC, CD , &c, and the middle instants of their pulses

(u) Sect. III. Art. 12.

(x) Def. II.

pulses (*y*) by the points *A, B, C, &c*; and the successive vibrations *v, v, v, &c.* by the equal lines *ab, bc, cd, &c.* and the middle instants of their pulses by the points *a, b, c, &c.*

Then beginning from two coincident pulses at *A* or *a*, it is observable, that the successive intervals of the pulses are alternately bigger and less; and that the Alternate Lesser Intervals *Bb, Cc, Dd, &c.* or $V-v, 2V-2v, 3V-3v, \&c.$ increase uniformly, by the repeated addition of the first lesser interval $V-v$, at every equal increment V or v of their distances from *A*. The alternate lesser intervals are therefore proportional to their distances from the coincident pulses *A, a*.

Now any assigned distance $3V : rV :: 3v : rv :: 3V-3v : rV-rv = dv$, by the equation; whence $3V : \frac{r}{d}V :: 3V-3v : v$; consequently if the assigned distance $3V$ or *AD* be less than half the simple cycle or period $\frac{r}{d}V$, the adjoining interval $3V-3v$, or *Dd* is less than half v ; but if bigger, bigger than half v .

And the argument is the same in going backwards from the two next coincident pulses at *U* and *w*, *U* and *x, &c.* their larger and lesser alternate intervals being evidently of the same magnitudes as in going forwards.

Fig. 24. Now if the difference $d=2$, and the length of the complex cycle be the line *AU* or $ax=rV=rv+2v$, having divided it into two equal

(*y*) Sect. I. Art. 12.

equal parts AX , XU , we have the part or period $AX = \frac{r}{2} V$; which because 2 does not measure r (z), consists of a multiple of V , as AK , and a remainder $KX = \frac{1}{2} V = \frac{1}{2} KL$.

We have also, by the same equation, $AX = \frac{r}{2} v + v$, which because 2 does not measure r , consists of a multiple of v (one more than that other of V) as Al , and a like remainder $lX = \frac{1}{2} v = \frac{1}{2} lm$.

Now the distances of the successive pulses of V from the point X are XL , XM , XN , &c, or $\frac{1}{2} V$, $\frac{3}{2} V$, $\frac{5}{2} V$, &c, and those of the successive pulses of v are Xm , Xn , Xo , &c, or $\frac{1}{2} v$, $\frac{3}{2} v$, $\frac{5}{2} v$, &c, and the differences of those respective distances, or the alternate lesser intervals between the successive pulses of V and v , are Lm , Mn , No , &c, or $\frac{1}{2} V - \frac{1}{2} v$, $\frac{3}{2} V - \frac{3}{2} v$, $\frac{5}{2} V - \frac{5}{2} v$, &c; which increase uniformly by the repeated addition of $V - v$ or $\frac{2}{2V - 2v}$ to the first and succeeding intervals.

Assign any distances XL , XN , or $\frac{1}{2} V$ and $\frac{5}{2} V$; then $\frac{1}{2} V : \frac{5}{2} V :: \frac{1}{2} v : \frac{5}{2} v :: \frac{1}{2} V - \frac{1}{2} v : \frac{5}{2} V - \frac{5}{2} v$, that is, $XL : XN :: Xm : Xo :: Lm : No$, or the alternate lesser intervals are proportional to their distances from the periodical point X .

Now

(z) For if 2 measured r , it would also measure $R = r + 2$, and so the terms R , r of the ratio V to v would not be the least, as they are supposed to be.

Now any assigned distance $\frac{r}{2}V : \frac{r}{2}V :: \frac{r}{2}v : \frac{r}{2}v$
 $v :: \frac{r}{2}V - \frac{r}{2}v : \frac{r}{2}V - \frac{r}{2}v = v$, by the given equation, that is, $\frac{r}{2}V : \frac{r}{2}V :: \frac{r}{2}V - \frac{r}{2}v : v$; consequently if the assigned distance $\frac{r}{2}V$ or XN , be less than half the period $\frac{r}{2}V$ or half XU , the adjoining interval $\frac{r}{2}V - \frac{r}{2}v$ or No , is less than half v ; but if bigger, bigger than half v .

And in going backwards from X , the alternate lesser intervals Kl , Ik , Hi , &c, are respectively equal to Lm , Mn , No , &c, at equal distances on each side of X .

Fig. 25. In like manner if $d=3$, or $rV=rV + 3v$, having divided this cycle AU or ay into three equal periods AX , XI , IU , that equation gives $AX = \frac{r}{3}V$, which consists of a multiple of V , as AG , and a remainder $GX = \frac{1}{3}V$ (or $\frac{2}{3}V$ hereafter to be considered) $= \frac{1}{3}GH$, whose complement $XH = \frac{2}{3}V$.

The same period AX is also $= \frac{r}{3}v + v$ by the same equation, and therefore consists of a multiple of v (one more than that other of V) as Ab , and a like remainder $bX = \frac{1}{3}v = \frac{1}{3}bi$, whose complement $Xi = \frac{2}{3}v$.

Hence the distances from X of the successive pulses of V are XH , XI , XK , &c, or $\frac{2}{3}V$, $\frac{5}{3}V$, $\frac{8}{3}V$, &c, and those of the successive pulses of v are Xi , Xk , xl , &c, or $\frac{2}{3}v$, $\frac{5}{3}v$, $\frac{8}{3}v$, &c. and their differences, or the alternate
 lesser

lesser intervals between the successive pulses of V and v , are Hi , Ih , Kl , &c, or $\frac{2}{3}V - \frac{2}{3}v$, $\frac{5}{3}V - \frac{5}{3}v$, $\frac{8}{3}V - \frac{8}{3}v$, &c; which increase uniformly by the repeated addition of $V - v$ or $\frac{3V - 3v}{3}$ to the first and succeeding intervals.

Assign any distances XH and XK , or $\frac{2}{3}V$ and $\frac{8}{3}V$; then $\frac{2}{3}V : \frac{8}{3}V :: \frac{2}{3}v : \frac{8}{3}v$, $:: \frac{2}{3}V - \frac{2}{3}v : \frac{8}{3}V - \frac{8}{3}v$, that is, $XH : XK :: Xi : Xl :: Hi : Kl$, or the alternate lesser intervals are proportional to their distances from the periodical point X .

Now any assigned distance $\frac{8}{3}V : \frac{r}{3}V :: \frac{8}{3}v : \frac{r}{3}v :: \frac{8}{3}V - \frac{8}{3}v : \frac{r}{3}V - \frac{r}{3}v = v$ by the equation, that is, $\frac{8}{3}V : \frac{r}{3}V :: \frac{8}{3}V - \frac{8}{3}v : v$; so that if the assigned distance $\frac{8}{3}V$ or XK be less than half the period $\frac{r}{3}V$ or half $X\mathcal{Y}$, the adjoining interval $\frac{8}{3}V - \frac{8}{3}v$ or Kl is less than half v ; but if bigger, bigger than half v .

By doubling the period $AX = AG + \frac{1}{3}V$, we have $AY = 2AG + \frac{2}{3}V = AN + \frac{2}{3}V$, so that NY is $\frac{2}{3}V$ and its complement $YO = \frac{1}{3}V$. Again by doubling $AX = Ab + \frac{1}{3}v$, we have $AY = 2Ab + \frac{2}{3}v = Ap + \frac{2}{3}v$, so that $py = \frac{2}{3}v$ and its complement $Yq = \frac{1}{3}v$.

Hence the alternate lesser intervals of the pulses of V and v , in going opposite ways to equal distances from X and from \mathcal{Y} , are equal. And in going contrary ways from X towards A , and from \mathcal{Y} towards U , the alternate lesser intervals

intervals are $\frac{1}{3}V - \frac{1}{3}v$, $\frac{4}{3}V - v\frac{4}{3}$, $\frac{7}{3}V - \frac{7}{3}v$, &c, which increase uniformly as before; and $\frac{7}{3}V$ being an assigned distance from X or γ , we have $\frac{7}{3}V : \frac{r}{3}V :: \frac{7}{3}v : \frac{r}{3}v :: \frac{7}{3}V - \frac{7}{3}v : \frac{r}{3}V - \frac{r}{3}v = v$ as before. So that if the assigned distance $\frac{7}{3}V$ be less than half the period $\frac{r}{3}V$, the adjoining interval $\frac{7}{3}V - \frac{7}{3}v$ is less than half v ; but if bigger, bigger than half v .

Fig. 26. Lastly when the period $AX = \frac{r}{3}V$, consists of a multiple of V as AG and a remainder $GX = \frac{2}{3}V$, which remained to be considered, its complement XH is $= \frac{1}{3}V$, and the demonstration would proceed in the same method as before.

Whoever desires a general proof of the proposition for any value of the difference d , need only read the last example over again with a design to make the proof general; and he will perceive that what has been said of the number 3 as a value of d , *mutatis mutandis*, is plainly applicable to any other value. Q. E. D.

Coroll. 1. Any simple cycle or period of the pulses of imperfect unisons contains one more of the quicker than of the slower vibrations, as appears by its equation, $\frac{r}{d}V = \frac{r}{d}v + v$; and the periodical points X , γ , &c, always fall within those values of v that are severally contained within as many corresponding values of V , and the number of those points in each complex cycle is $d - 1$.

Coroll.

Coroll. 2. The lesser intervals that lie nearest to the periodical points and the points of coincidence, are less than any of the rest and are $\frac{V-v}{d}$ and all its multiples, whereof the greatest multiplier is d ; as $\frac{V-v}{3}$, $\frac{2V-2v}{3}$, $\frac{3V-3v}{3}$, when $d=3$; $\frac{V-v}{4}$, $\frac{2V-2v}{4}$, $\frac{3V-3v}{4}$, $\frac{4V-4v}{4}$, when $d=4$; &c.

Coroll. 3. Some of the alternate lesser intervals of the pulses of imperfect unisons, are the differences of equal numbers of their vibrations, counted from the nearest coincident pulses; and others are the differences of equal numbers of the same part or parts of their single vibrations, counted from the nearest periodical point.

Coroll. 4. If the vibrations of two couples of imperfect unisons, or of any two consonances, be proportional, the periods and cycles of their pulses, whether simple or complex, will be in the ratio of the homologous vibrations.

Let T and t be the vibrations of one couple, and V and v those of the other; and since $T : t :: V : v :: r+d : r$, the cycles of their pulses are $rT = \overline{r+d} \times t$ and $rV = \overline{r+d} \times v$, and the periods are $\frac{r}{d} T = \frac{r+d}{d} t$ and $\frac{r}{d} V = \frac{r+d}{d} v$; and are in the ratio of T to V , or of t to v .

Coroll. 5. The length of the period of the Least Imperfections in any consonance of imperfect unisons, is the same as that of the period of its pulses.

Pl. XI. Fig. 23, 24, 25, 26. For unisons are perfect when their successive pulses are constantly coincident (*a*), and imperfect when the ratio of their vibrations is a little altered from the ratio of equality (*b*); and then the pulses are gradually separated by Alternated Lesser Intervals, which are the Imperfections of this consonance; and since they increase in going from the beginning to the middle of every simple cycle, or period of the pulses, and thence decrease to the end of it (*c*), the length of the period of the Least Imperfections of imperfect unisons is plainly the same as that of the period of their pulses.

PROPOSITION VIII.

If either of the vibrations of imperfect unisons and any multiple of the other, or any different multiples of both, whose ratio is irreducible, be considered as the single vibrations of an imperfect consonance, the length of the period of its least imperfections, will be the same as that of the pulses of the imperfect unisons.

Pl. XI. Fig. 23, 27. For instance, if AB and ab be the vibrations of imperfect unisons,
 $2AB$

(*a*) Sect. I. Art. 3.
 (*c*) Prop. VII.

(*b*) Sect. VI. Defin. I.

$2AB$ or AC and ab will be the vibrations of imperfect octaves, whose treble is one of the unisons, and whose base is derived from the other by intermitting every other pulse of the series, A, B, C, D, E , &c.

Now if these octaves were perfect, every pulse of the base would coincide with every other pulse of the treble; but here they are gradually separated by some of the alternate lesser intervals Cc, Ee , &c, of the imperfect unisons. The intermediate pulses of the treble, which in perfect octaves would bisect the intervals of the pulses of the base, are also gradually separated from the round points which bisect them, by the rest of the alternate lesser intervals of the said unisons. And thus the imperfections of the tempered octaves, or the Dislocations of the pulses in their successive short cycles (d), are every where the same as the imperfections of the unisons, and consequently have the same periods.

The argument is the same if $2ab$ or ac and AB be the vibrations of imperfect octaves, as in Fig. 28; and also if any other multiple of AB or ab , as mAB or mab , be one of the vibrations of the imperfect consonance; as appears by supposing $m-1$ pulses of AB or ab to be so intermitted as to leave only single equidistant pulses in Fig. 23, 24, 25, 26.

Pl. XII. Fig. 34. Now let any different multiples of AB and ab , as $3AB$ and $2ab$, or

E

AD

(d) Defn. IV. Sect. VI.

AD and ac be the vibrations of the imperfect consonance; and if AB were $= ab$, then would $3AB$ or AD be to $2ab$ or $ac :: 3 : 2$, and all the short cycles of the vibrations, AD , ac would be perfect, or their exterior pulses G and g , N and n , &c, would be coincident, as in Fig. 33 : because $2 \times 3AB$ or $2AD$ or $AD + DG$ would then $= 3 \times 2ab$ or $3ac$ or $ac + ce + eg$.

But AB in Fig. 34 being bigger than ab , the multiple $6AB$ or AG is also bigger than the equimultiple $6ab$ or ag ; and so the Exterior pulses G and g , N and n , &c, of the short cycles, which pulses were coincident before, are now separated by some of the alternate lesser intervals, Gg , Nn , &c, of the pulses of AB and ab (e): the distances of the pulses G and g , N and n , &c, from A and a , being equimultiples of AB and ab .

For the like reason the Interior pulses, c , e , &c, of the imperfect short cycles are also separated from the pulses of AB and ab (denoted by round points when different from those of AD and ac) by some of their alternate lesser intervals.

Hence the dislocations of the pulses in all the imperfect short cycles, are some of the alternate lesser intervals of the pulses of Ab and ab .

For though we began the first short cycle from two coincident pulses A, a , yet the argument is the same if we suppose them separated by

by any one of the alternate lesser intervals; or begin to count the vibrations of the consonance from any two pulses of AB and ab , as Q and r , whose distances from the next periodical point or coincident pulses Z , n , are proportional to the vibrations AB , ab , that is, whose interval Qr is an alternate lesser interval of their pulses (*f*).

For since the several lengths QX and ry , $X\Delta$ and $y\epsilon$, ΔK and $\epsilon\lambda$, &c, of the subsequent short cycles, are proportional to AB and ab , the remaining distances XZ and yn , ΔZ and ϵn , KZ and λn , &c, are also proportional to AB and ab ; which shews that the dislocations of the exterior pulses X and y , Δ and ϵ , K and λ , &c, and of the interior too, are constantly some of the alternate lesser intervals of the pulses of AB and ab . And thus the period of the least dislocations of the pulses of the imperfect consonance, or of the least imperfections in its short cycles, is constantly the same as that of the pulses of AB and ab .

Fig. 35 shews the same thing, when $2AB$ and $3ab$, or AC and ad , are different multiples of AB and ab , whose pulses make long simple cycles; and when they make periods, the like is evident by inspection of the pulses about the periodical points X, Y , in Fig. 24, 25, 26, supposing the proper numbers of pulses to be intermitted.

And universally, the least terms of any perfect ratio being m and n , the periods of im-

E 2

perfect

(*f*) Prop. VII.

perfect unisons whose vibrations are AB and ab , will be changed into periods of the same length of an imperfect sharp consonance whose vibrations are mAB and nAb by intermitting $m-1$ pulses of AB and $n-1$ pulses of ab ; or into equal periods of an imperfect flat consonance whose vibrations are mab and nAB , by intermitting $m-1$ pulses of ab and $n-1$ pulses of AB , so as to leave equidistant pulses at larger intervals for the pulses of the resulting consonance.

For tho' some of the alternate lesser intervals of the pulses of the imperfect unisons are destroyed by those intermissions, yet the remaining pulses continue in their own places and make periods of the same length as the whole number of pulses did before. Q. E. D.

Coroll. 1. With respect to the perfect consonance whose vibrations are mAB , and nAB , the former imperfect consonance of mAB and nab is tempered sharp by the tempering ratio nAB to nab (g), and the latter imperfect consonance of mab and nAB is tempered flat by the same ratio mAB to mab of the vibrations AB , ab of the imperfect unisons, whose interval is therefore the temperament of both those imperfect consonances.

And the same might be said with respect to this other perfect consonance of the vibrations mab and nab , whose interval is the same as that of the former perfect consonance, the perfect ratio being the same in both.

Coroll.

(g) Sect. II. Art 5 and 6.

Coroll. 2. The lengths of the perfect cycles of those perfect consonances are $mn AB$ and $mnab$; (because $mAB : nAB :: m : n :: mab : nab$;) and $mnAB$ being the greater of the two, is therefore the whole length of the imperfect short cycle of either of the foregoing tempered consonances.

Coroll. 3. Consequently the imperfect short cycle of any imperfect consonance contains equal numbers of the slower and quicker vibrations AB , ab of the imperfect unisons from whence it is derived.

Coroll. 4. The same multiples of the vibrations of imperfect unisons, will be the vibrations of other imperfect unisons, whose period is the same multiple of the period of the given unisons (b), and whose interval is the same too at a different pitch; because the ratio of the vibrations is the same (i).

L E M M A.

Pl. XII. Fig. 36. *The logarithms of small ratios, a o to b o, c o to d o, whose terms have a common half sum, so, are very nearly proportional to the differences of the terms of each ratio.*

For by the supposition the point s bisects both ab and cd , the differences of those terms. And if any hyperbola described with the center o and

E 3 rectangular

(b) Prop. VII. coroll. 4. (i) Sect. I. Art. 10.

rectangular asymptotes so, oq , cuts the perpendiculars erected at s, a, b, c, d , in t, e, f, g, h , it is well known that the areas $abfe, cdbg$, arithmetically expressed, are hyperbolic logarithms of the ratios ao to bo , co to do ; and that these logarithms are proportional to any other logarithms of the same ratios. And those areas $abfe, cdbg$, differ very little from the trapeziums $ablk, cdnm$, cut off by the tangent ptq at the point t in the middlemost perpendicular ts : because the common bases ab, cd , or differences of the terms ao and bo , co and do , are supposed to be very small in comparison to the terms themselves.

It appears then that the logarithm of the ratio ao to bo is to the logarithm of co to do , as the area $abfe$ to the area $cdhg$, or very nearly as the trapezium $ablk$ to the trapezium $cdnm$, or (because st is the mean altitude of both) as the rectangle under ab and st to the rectangle under cd and st , or as ab to cd . Q. E. D.

Coroll. 1. The logarithms of small ratios, ao to bo , ao to co , which have a common term ao , are also very nearly proportional to the differences of their terms; but not so nearly as if the terms had a common half sum.

For the logarithms of the ratios ao to bo , ao to co are proportional to the areas $abfe, acge$, or very nearly to the trapeziums $ablk, acmk$, or to the rectangles under their bases ab, ac and their mean altitudes, or nearly to the bases themselves: because the ratio of the mean altitudes is very small in comparison to that of the bases.

Coroll.

Coroll. 2. Pl. XII. Fig. 37. The logarithms of any small ratios $ao : bo$, $co : do$ are very nearly in the ratio of $\frac{ab}{ao}$ to $\frac{cd}{co}$, or of $\frac{ab}{bo}$ to $\frac{cd}{do}$, compounded of the direct ratio of the differences of the terms of the proposed ratios, and the inverse ratio of their homologous terms.

For supposing $ao : eo :: co : do$, these ratios have the same logarithm. Whence the logarithms of the proposed ratios, $ao : bo$, $co : do$, or $ao : eo$, are as ab to ae by cor. 1, or as $\frac{ab}{ao}$ to $\frac{ae}{ao}$ or $\frac{cd}{co}$, because $ae : ao :: cd : co$ by the supposition. The second part may be proved in like manner by taking $fo : ao :: co : do$.

Coroll. 3. If a and b be the terms of any small ratio whose logarithm is c and $\frac{q}{p}c$ be any part or parts of it; taking $s = \frac{a+b}{2}$ and $d = \frac{a-b}{2}$, $s + \frac{q}{p}d$ to $s - \frac{q}{p}d$ is the ratio whose logarithm is $\frac{q}{p}c$ very nearly.

For the terms of both those ratios have a common half sum s , and since $s + d = a$ and $s - d = b$, the difference of the terms a and b is $2d$, and that of the terms $s + \frac{q}{p}d$ and $s - \frac{q}{p}d$ is $\frac{2q}{p}d$. Whence by the lemma, the logarithm of a to b is to the logarithm of $s + \frac{q}{p}d$ to $s - \frac{q}{p}d$ $:: 2d : \frac{2q}{p}d :: 1 : \frac{q}{p} :: c : \frac{q}{p}c$, and c being the logarithm

rithm of a to $b, \frac{q}{p} c$ is the logarithm of $s + \frac{q}{p} d$ to $s - \frac{q}{p} d$,

Coroll. 4. Hence as musical intervals are proportional to the logarithms of the ratios of the single vibrations of the terminating sounds (k), if any part or parts of a comma c denoted by $\frac{q}{p} c$, be the interval of imperfect unisons, the ratio of the times of their single vibrations will be $161p+q$ to $161p-q$.

For the comma c being the interval of two sounds whose single vibrations are as 81 to 80 (l), by substituting 81 for a and 80 for b in the last corollary, we have $s = \frac{161}{2}$, $d = \frac{1}{2}$ and $s + \frac{q}{p} d$ to $s - \frac{q}{p} d :: \frac{161}{2} + \frac{q}{p} \times \frac{1}{2} : \frac{161}{2} - \frac{q}{p} \times \frac{1}{2} :: 161p+q : 161p-q$, the ratio of the single vibrations belonging to the interval $\frac{q}{p} c$, very nearly (m).

This

(k) Sect. I. Art. 11. (l) Sect. II. Art. 4.

(m) And conversely, if the ratio of the times of the single vibrations of imperfect unisons be V to v , their interval is $\frac{V-v}{V+v} \times 161c$. For supposing $V : v :: 161p+q : 161p-q$, p and q being indeterminate numbers; or $V = 161p+q$ and $v = 161p-q$; then $V-v = 2q$ and $V+v = 161 \times 2p$, and $\frac{V-v}{V+v} = \frac{q}{161p}$. Whence $\frac{V-v}{V+v} \times 161c = \frac{q}{p} c$, the interval belonging to the ratio $161p+q : 161p-q$, or $V : v$, by coroll. 4.

This ratio approaches surprisngly near to the truth, as will appear by an example. Let $\frac{q}{p}c = \frac{1}{4}c$, or $p=4$ and $q=1$, then $161p+q : 161p-q ::$ as $645 : 643$. Now by the Tables of Logarithms,

$$\text{The log. } \frac{81}{80} = 0.00539.50319$$

$$\frac{1}{4} \text{ of it} = 0.00134.87580$$

$$\text{the log. } \frac{645}{643} = 0.00134.87417$$

$$\text{the difference} = 0.00000.00163$$

and the logarithm 539.50319 divided by the difference 163 gives the quotient 330984, which shews that $\frac{1}{4}c$ deduced from the ratio 645:643 differs from the truth but by $\frac{1}{330984}$ part of a comma; a degree of exactness abundantly sufficient for every purpose in harmonics.

Coroll. 5. The times of the single vibrations of imperfect unisons being V and v and their interval $\frac{q}{p}c$; $V : v :: 161p+q : 161p-q$ and the period of their pulses is $\frac{161p-q}{2q} V$ or $\frac{161p+q}{2q} v$ (n).

Likewise the Vibrations of other imperfect unisons being V' and v' and their interval $\frac{q'}{p'}c$; $V' : v' :: 161p+q' : 161p-q'$ and the period of their pulses is $\frac{161p-q'}{2q'} V'$ or $\frac{161p+q'}{2q'} v'$.

Coroll.

(n) See Sect. VI. Defin. III.

Coroll. 6. Hence if the intervals of two consonances of imperfect unisons be equal, or $q=q'$, the periods of their pulses have the ratio of their slower or quicker vibrations, V to V' , or v to v' , which ratios are therefore the same (o).

Coroll. 7. The Ultimate Ratio of the periods of imperfect unisons is compounded of the ratio of their slower or quicker vibrations directly and of that of their intervals inversely, and so it is $\frac{V}{q}$ to $\frac{V'}{q'}$, or $\frac{v}{q}$ to $\frac{v'}{q'}$.

For supposing the quantities p , v , v' in coroll. 5. to be variable and p to increase to infinity in any finite time, the intervals $\frac{q}{p}c$, $\frac{q'}{p}c$, will decrease and vanish in the ratio of q to q' first given; the ratio $161 p - q$ to $161 p - q'$ will also decrease and vanish in the ratio of equality; and therefore the Ultimate Ratio with which the increasing periods $\frac{161 p - q}{2 q} V$ and $\frac{161 p - q'}{2 q'} V'$ became infinite at the end of the given time, and vanished into innumerable short cycles of perfect unisons, is $\frac{V}{q}$ to $\frac{V'}{q'}$, or, by a like argument, $\frac{v}{q}$ to $\frac{v'}{q'}$.

Coroll. 8. Hence if two consonances of imperfect unisons have a common sound or vibration $V=V'$, or $v=v'$, the ultimate ratio of their periods is q' to q , the inverse ratio of their intervals;

(o) This agrees with Cor. 4. Prop. VII.

intervals ; and consequently is the inverse ratio of the differences of their vibrations (p).

Coroll. 9. If the two flower or the two quicker vibrations of two consonances of imperfect unisons have the ratio of their intervals, the periods of their pulses are ultimately equal.

For if $V : V' :: q : q'$, then $\frac{V}{q} : \frac{V'}{q'} :: 1 : 1$, which is the ultimately ratio of the periods : and the like argument is applicable to the ratio $v : v'$.

PROPOSITION IX.

If the interval of two sounds whose perfect ratio is m to n, be increased or diminished by $\frac{q}{p}c$, and the times of the complete vibrations of the base and treble of either of these consonances be Z and z and the period of its least imperfections be P, then in

$$\text{Cas. 1, } P = \frac{161p-q}{2q} \times \frac{Z}{m} \text{ or } \frac{161p+q}{2q} \times \frac{z}{n},$$

$$\text{Cas. 2, } P = \frac{161p+q}{2q} \times \frac{Z}{m} \text{ or } \frac{161p-q}{2q} \times \frac{z}{n}.$$

Pl. XII. Fig. 38. For if AV and av, or V and v be the times of the complete vibrations of imperfect unisons whose interval is the temperament $\frac{q}{p}c$, then $V : v :: 161p+q : 161p-q$ (q) and the period of their pulses, or of their least imper-

(p) Coroll. 1. and Sect. 1. Art. II,

(q) Coroll. 4. Lemma.

imperfections, is $\frac{161p-q}{2q} V = \frac{161p+q}{2q} v$ (r) and has the same length as the period of the least imperfections in a sharp consonance whose vibrations are mV and nv , or in a flat consonance whose vibrations are mv and nV ; both consonances being derived from the perfect one whose vibrations are mV and nV , or mv and nv (s).

Hence in Caf. 1. taking $Z=mV$ and $z=nv$, we have $\frac{Z}{m}=V$ and $\frac{z}{n}=v$; which values being substituted for V and v in the period of imperfect unisons, give $P = \frac{161p-q}{2q} \times \frac{Z}{m} = \frac{161p+q}{2q} \times \frac{z}{n}$. And in Caf. 2. (Z and z being supposed indeterminate) taking $Z=mv$ and $z=nV$, we have $\frac{Z}{m}=v$ and $\frac{z}{n}=V$; which being substituted for v and V in the said period of imperfect unisons, give $P = \frac{161p+q}{2q} \times \frac{Z}{m} = \frac{161p-q}{2q} \times \frac{z}{n}$. Q. E. D.

The value of P in Case 2 is deducible from its value in Case 1, only by changing the sign of q ; that is, by supposing $\frac{q}{p}c$ to be the negative or flat temperament, as it really is when $+\frac{q}{p}c$ is the sharp one. And thus one expression of the value of P might have served both cases of the proposition, but two are more distinct for future use.

Coroll.

(r) Sect. VI. Defin. III. or Cor. 5: Lemma.

(s) Prop. VIII. Corol. I.

Coroll. 1. Hence if any two imperfect consonances have Z and Z' for the times of the complete vibrations of their bases, z and z' for those of their trebles, $\frac{q}{p}c$ and $\frac{q'}{p}c$ for their temperaments, whether sharp or flat, or one of each sort, m and m' for the major, and n and n' for the minor terms of the perfect ratios; the Ultimate ratio of their periods is $\frac{Z}{qm}$ to $\frac{Z'}{q'm'}$, or $\frac{z}{qn}$ to $\frac{z'}{q'n'}$: The proof of which is the same as was given for the Ultimate ratio of the periods of imperfect unisons in Coroll. 7. to the Lemma.

Coroll. 2. Hence when the temperaments are equal and the major terms the same, the periods of the least imperfections have ultimately the ratio of the single vibrations of the bases.

Coroll. 3. When the bases are the same, the periods have ultimately the inverse ratio of the temperaments and major terms jointly.

Coroll. 4. When the bases and major terms are the same, the periods have ultimately the inverse ratio of the temperaments.

Coroll. 5. When the bases and temperaments are the same, the periods have ultimately the inverse ratio of the major terms.

Coroll. 6. All those corollaries are applicable to the trebles and the minor terms, only by reading trebles instead of bases and minor terms instead of major; and then, as before they had no dependence on the trebles and minor terms,
so

so now they have none upon the bases and major terms.

PHÆNOMENA OF BEATS.

If a consonance of two sounds be uniform, without any beats or undulations, the times of the single vibrations, of its sounds have a perfect ratio; but if it beats or undulates, the ratio of the vibrations differs a little from a perfect ratio, more or less according as the beats are quicker or slower.

Change the first and smallest string of a violoncello for another about as thick as the second, that their sounds having nearly the same strength may beat stronger and plainer. Then skrew up the first string; and while it approaches gradually to an unison with the second, the two sounds will be heard to beat very quick at first, then gradually slower and slower, till at last they make an uniform consonance without any beats or undulations. At this juncture either of the strings struck alone, by the bow or finger, will excite large and regular vibrations in the other, plainly visible to the eye; which shews that the times of their single vibrations are equal (t).

Alter the tension of either string a very little, and their sounds will beat again. But now the
motion

(t) Sect. I. Art. I.

motion of one string struck alone makes the other only start, but excites no regular vibrations; a plain proof that they are not isochronous. And while the sounds of both are drawing out with an even bow, not only an audible but a visible beating and irregularity is observable in the vibrations, which in the former case were free and uniform.

Measure the length of either string between the nut and bridge, and, when they are perfect unisons, at the distance of $\frac{1}{3}$ of that length from the nut mark that string with a speck of ink. Then placing the edge of your nail on the speck, or very near it, and pressing it to the finger-board, upon sounding the remaining $\frac{2}{3}$ with the other string open, you will hear an uniform consonance of V^{th} s, whose single vibrations have the perfect ratio of 3 to 2 (*u.*) But upon moving your nail a little downwards or upwards, that ratio will be a little increased or diminished; and in both cases the imperfect V^{th} s will beat quicker or slower according as that perfect ratio is more or less altered.

The Phænomena are the same when the parts of the string have any other perfect ratio; except that the beats of the simpler concords are plainer than those of the less simple and these plainer than those of the discords, which being very quick are not easily distinguished from the uniform roughness of perfect discords.

The

The sounds of an organ being generally more uniform than any other, their beats are accordingly more distinct, and are perfectly isochronous when the blast of the bellows is so uniform as not to alter the vibrations of either sound.

Beats and undulations when every thing else is silent, are also pretty plain upon the harpsichord, especially while the sounds are vanishing.

Quicker undulations are beats, and are remarkably disagreeable in a concert of strong, treble voices, when some of them are out of tune; or in a ring of bells ill tuned, the hearer being near the steeple; or in a full organ badly tuned: nor can the best tuning wholly prevent that disagreeable battering of the ears with a constant rattling noise of beats, quite different from all musical sounds, and destructive of them, and chiefly caused by the compound stops called the Cornet and Sesquialter, and by all other loud stops of a high pitch, when mixed with the rest. But if we be content with compositions of unisons and octaves to the Diapason, whatever be the quality of their sounds, the best manner of tuning will render the noise of their beats inoffensive if not imperceptible. These are the general phenomena of beats, whose theory I am going to explain.

P R O-

PROPOSITION X.

An imperfect consonance makes a beat in the middle of every period of its least imperfections, and so the time between its successive beats is equal to the periodical time of its least imperfections.

Pl. XI. Fig. 23 to 27. 34, 35. Any simple cycle or any period of the pulses of imperfect unisons, contains one more of the quicker than of the slower vibrations (x), and the short cycle of any imperfect consonance contains equal numbers of the quicker and slower vibrations of the imperfect unisons (y). Consequently after taking away the greatest equal numbers of short cycles, that can be taken from both ends of the simple cycle or the period of the imperfect unisons, some part of another short cycle or two, as consisting of unequal numbers of the quicker and slower vibrations of the imperfect unisons, will always remain in the middle of the cycle or period. And this part, by interrupting the succession of the short cycles, wherein the harmony of the consonance consists, interrupts its harmony and causes the noise which is called a beat: especially as the interruption is made where the short cycles on each

(x) Prop. VII. coroll. 1.

(y) Prop. VIII. coroll. 3.

sive of it are the most imperfect and inharmonious. Therefore the time between the successive beats, made in the middle of each period or simple cycle of the pulses of the imperfect unisons, or of the least imperfections of the consonance (z), is equal to the time of this period.

And the cause of the beats of imperfect unisons is a like interruption of the succession of their short cycles, in the middle of every period or simple cycle of their pulses, where they are most imperfect and inharmonious. Q. E. D.

Coroll. The time between the successive beats of an imperfect consonance is the same as the periodical time of its Greatest Imperfections.

PROPOSITION XI.

If the interval of two sounds whose perfect ratio is m to n , be increased or diminished by the temperament $\frac{q}{p}c$ (a), and β be the number of beats made by either of those consonances while its base is making N , and its treble M complete vibrations; then in

$$\text{Cas. 1, } \beta = \frac{2q}{161p-q} mN, \text{ or } \frac{2q}{161p+q} nM,$$

$$\text{Cas. 2, } \beta = \frac{2q}{161p+q} mN, \text{ or } \frac{2q}{161p-q} nM.$$

For if the time between the successive beats of either consonance be called P , and the time of

(z) Prop. VIII. (a) See Lemma cor. 4. p. 72.

of a complete vibration of its base be Z and that of its treble z ; the time of their beating and vibrating will be constantly measured by $\beta P = NZ$ or Mz . Hence $\beta = \frac{NZ}{P}$ or $\frac{MZ}{P}$ and since the time P is equal to the period of the least imperfections of the consonance (b), by substituting its values in Prop. IX, we have in

Cas. 1. $\beta = NZ \times \frac{2qm}{161p-q} \cdot \frac{1}{Z} = \frac{2qmN}{161p-q}$, and so of the other values of β . Q. E. D.

Coroll. 1. Hence if any two imperfect consonances have Z and Z' for the times of the single vibrations of their bases, z and z' for those of their trebles, $\frac{q}{p}c$ and $\frac{q'}{p'}c$ for their temperaments, whether flat or sharp, or one of each sort, m and m' for the major, n and n' for the minor terms of the perfect ratios, N and N' for the numbers of complete vibrations made by the bases, and M and M' for those made by the trebles in any given time; the ultimate ratio of the numbers of their beats, made in that time, will be $qmN : q'm'N'$, or $qnM : q'n'M'$, or $\frac{qm}{Z} : \frac{q'm'}{Z'}$, or $\frac{qn}{z} : \frac{q'n'}{z'}$.

The manner of proving the two first ratios has been shewn before (c), and the given time being constantly $NZ = N'Z' = Mz = M'z'$, we have $N : N' :: \frac{1}{Z} : \frac{1}{Z'}$, which ratios compounded with $qm : q'm'$ give $qmN : q'm'N' :: \frac{qm}{Z} : \frac{q'm'}{Z'}$.

F 2

Like-

(b) Prop. x.

(c) Lemma cor. 7.

Likewise $M : M' :: \frac{1}{z} : \frac{1}{z'}$ which ratios compounded with $qn : q'n'$ give $qnM : q'n'M' :: \frac{qn}{z} : \frac{q'n'}{z'}$.

Coroll. 2. Hence, when the temperaments are equal and the major terms the same, the beats of the consonances, made in a given time, have ultimately the inverse ratio of the single vibrations of the bases.

Coroll. 3. When the bases are the same, the beats have ultimately the ratio of the temperaments and major terms jointly: And therefore when the bases and beats are the same, the temperaments have ultimately the inverse ratio of the major terms.

Coroll. 4. When the bases and major terms are the same, the beats have ultimately the ratio of the temperaments.

Coroll. 5. When the bases and temperaments are the same, the beats have ultimately the ratio of the major terms.

Coroll. 6. All these corollaries are applicable to the trebles and minor terms, by reading trebles instead of bases and minor terms instead of major: and then they have no dependence on the bases and major terms, as in the former cases they had none upon the trebles and minor terms: which absent terms may therefore in both cases have any magnitudes whatever without altering the ratio of the beats.

Coroll.

Coroll. 7. Things remaining as in the proposition, we have in

$$\text{Cas. 1. } Z : z :: m + \frac{\beta}{N} : n :: m : n - \frac{\beta}{N}.$$

$$\text{Cas. 2. } Z : z :: m - \frac{\beta}{N} : n :: m : n + \frac{\beta}{N}.$$

For by the Prop. in Cas. 1. $\beta = \frac{2qmN}{161p-q}$, whence $\frac{\beta}{N} : m :: 2q : 161p - q$ and *compositè* $m + \frac{\beta}{N} : m :: 161p + q : 161p - q$, either of which ratios being the tempering ratio and m to n the perfect one, the imperfect ratio is plainly $m + \frac{\beta}{N} : n :: Z : z$. And a like resolution of the other values of β in the proposition gives the other proportions.

Scholium 1.

To shew that the ultimate ratio of the beats or the periods of imperfect consonances, when used instead of the exact ratio, can produce no sensible difference in the Harmony.

1. The temperaments of any two consonances being $\frac{q}{p}c$ and $\frac{q'}{p}c$, the difference between the exact and the ultimate ratio of their beats, made in any given time, is the ratio $161p \mp q$ to $161p \mp q'$; where the sign of q or q' is negative if the respective temperament be sharp, or affirmative if flat.

For that ratio compounded with the exact ratio of the beats, which is $\frac{q m N}{161 p \mp q}$ to $\frac{q' m' N'}{161 p \mp q'}$ (*d*), makes their ultimate ratio $q m N$ to $q' m' N'$ (*e*).

2. Now the magnitude of the ratio $161 p \mp q$ to $161 p \mp q'$, like that of all ratios, being greatest or least according as the difference of its terms is greatest or least in proportion to the terms themselves; it will follow; that in the most harmonious system of sounds hereafter determined (*f*), the ultimate ratio of the beats of any two concords cannot differ from their exact ratio by any ratio greater than $362\frac{5}{8}$ to $361\frac{5}{8}$, or less than 2901 to 2900.

For the temperaments $\frac{q}{p} c$, $\frac{q'}{p} c$ of any two concords in that system, have no other values than a couple of these, $\frac{5}{18} c$, $\frac{2}{18} c$, and $\frac{1}{18} c$ (*f*). Where p being 18, the greatest magnitude of the said ratio $161 p \mp q$ to $161 p \mp q'$ is $161 \times 18 + 3$ to $161 \times 18 - 5$, or $362\frac{5}{8}$ to $361\frac{5}{8}$, and the least magnitude of it is $161 \times 18 + 3$ to $161 \times 18 + 2$, or 2901 to 2900.

3. Hence the number of beats in either term of any ultimate ratio in that system, cannot differ from the number of them in the corresponding term of the exact ratio, by above $\frac{1}{361}$ part of that first number: and therefore not by

(*d*) Prop. XI.

(*e*) Prop. XI. cor. I.

(*f*) Prop. XVI. Schol. 2. Art. 10 and 13.

by a single beat when that number is less than 361.

For let a to e be an ultimate ratio which exceeds the corresponding exact ratio by the greatest difference $362\frac{5}{8}$ to $361\frac{5}{8}$. Then by subtracting this difference, and neglecting the fractions, the exact ratio is $361a$ to $362e$, that is, a to $e + \frac{1}{361}e$, or $a - \frac{1}{362}a$ to e .

4. Now let two v^{ths} , or any two concords of the same name, near the middle of the scale of a good organ, have the same base and different trebles; and suppose them so nicely tempered, that in a given time one of the v^{ths} shall make 362 beats and the other 361. This indeed is extremely difficult to execute, the numbers of beats being so large. But supposing it done, my opinion is (from my own experience in smaller numbers) that the most critical ear could not distinguish the least difference in the harmony of those v^{ths} , or in the rate of their beating: no not if the ratio of the beats were much greater than 362 to 361: And if it could not, without doubt the theory of ultimate ratios is sufficiently accurate for determining and adjusting the Harmony of the best system of sounds. Because it will be shewn hereafter, that the best method of tuning any system, is to adjust every v^{th} to the number of beats it should make in that system.

5. In less harmonious systems, the difference between the exact and the ultimate ratio is some-

thing greater than 362 to 361 ; as $322\frac{1}{2}$ to $321\frac{1}{2}$ in the system of mean tones (*g*); but still not so great in any tolerable system as to affect the most critical ear: and what has been proved of beats holds true of Periods, the ratio of the periods being the inverse ratio of the numbers of beats made in any given time.

6. Therefore the ultimate ratios of beats and periods ought to be used in harmonics, their terms being always simpler than those of the exact ratios, as appears by comparing Prop. IX and XI with their corollaries.

For instance in the system of equal harmony, the temperament of the v^{th} is $\frac{-5}{18}c$, and of the vi^{th} is $\frac{+3}{18}c$, whence if their bases be the same the exact ratio of their beats, made in any given time, is $361\frac{7}{8}$ to $362\frac{7}{8}$ by Prop. XI; but their ultimate ratio is that of equality by coroll. 3, which is simpler, and the harmony of the concords not sensibly different (*b*).

Scholium 2.

To shew that the theory of beats agrees with experiments.

1. Pl. 1. Fig. 3. The exponent of the time of a single vibration of any given sound, as c ,
in

(*g*) Prop. 2. coroll.

(*b*) Art. 4. of these.

in a given system of perfect consonances may be changed into 1 by dividing every exponent by that of the given sound, which changes them to those in Pl. XII. Tab. I. without altering their proportions.

Then if the sound whose exponent is 1, be a little altered to γ either higher or lower, the numbers of beats made in any given time by the several imperfect consonances of γ with every one of the other sounds, will be proportional to the Denominators of their exponents.

For when γ is flatter than c , all the intervals above c are increased by the common temperament $c\gamma = \frac{2}{p}c$ in Prop. XI, where in Case 1 the number of beats made by any given consonance γd , while its base γ is making N vibrations, is $\frac{2q}{161p-q} m N$. And all the perfect intervals below c being diminished by $c\gamma$, in Case 2 the number of beats of any given consonance γB , while its treble γ is making M vibrations, is $\frac{2q}{161p-q} n M$.

Here the numbers N , M of the vibrations of γ made in any given time are equal, and $\frac{2q}{161p-q}$ being the same in both cases, the beats of γd are to those of γB as m to n , that is as the major term 9 of the perfect ratio 9 to 8 belonging to

to γd , is to the minor term 15 of the perfect ratio 15 to 16 belonging to γB ; and those terms are the Denominators of the exponents of d and B , the treble of the former and the base of the latter consonance.

And since the beats of γd and γB are as 9 to 15, and by the same demonstration those of γB and γe as 15 to 5, *ex æquo* the beats of γd and γe , having the same base, are as 9 to 5; which terms are the Denominators of the exponents of the trebles d, e .

And by the like proportions the beats of $\gamma B, \gamma A$, which have the same treble, are as 15 to 5, the Denominators of the exponents of the bases B, A .

And when γ is sharper than c , the two theorems above are changed to these $\frac{2q}{161p+q} m N$ and $\frac{2q}{161p+q} n M$, and the demonstration goes on as before. Q. E. D.

2. In Tab. 2 and 3 each series of fractions, being a geometrical progression in the ratio 2 to 1, are the exponents of the single vibrations of successive VIII^{ths}, and are severally deduced from the exponents of the bases of as many given concords AC, AD, AF, AC✕, AE, AF✕.

Hence in Tab. 2. the beats which the treble of any imperfect Minor consonance, AC, AD or AF, makes in a given time with its base and with every 8th below it and as many 8^{ths} above it

it as result from a continual bisection of the Numerator of its exponent, are all isochronous. But the beats which that treble makes with the successive 8^{ths} still higher are continually doubled in any given time.

T A B. 2.

$\frac{12}{5}$	$\frac{6}{5}$	1	$\frac{3}{5}$	$\frac{3}{10}$	$\frac{3}{20}$
A'	A	C	<i>a</i>	<i>a'</i>	<i>a''</i>
$\frac{8}{3}$	$\frac{4}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{6}$
A'	A	D	<i>a</i>	<i>a'</i>	<i>a''</i>
$\frac{16}{5}$	$\frac{8}{5}$	1	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{5}$
A'	A	F	<i>a</i>	<i>a'</i>	<i>a''</i>

For the major term of the perfect ratio of any Minor consonance is an Even number (*i*) and is the Numerator of the exponent of its base (that of its treble being reduced to 1,) and when that numerator is reduced to an odd number by continual bisections, this odd number is the constant numerator of the exponents of all the superior 8^{ths}, whose denominators must therefore be continually doubled, which doubles the beats by Art. 1. But the doubling the numerators of the exponents of the inferior 8^{ths} alters not their given denominator, as being an odd number, nor consequently the beats.

Tab. 3.

(*i*) Sect. 2. Art. 1. Table.

Tab. 3. The beats which the treble of any imperfect Major consonance, AC※, AE or AF※, makes with its base, in any given time, and with every 8th above it and as many 8^{ths} below it as result from a continual bisection of the Denominator of its exponent, if an Even number, are continual proportionals in the ratio of 2 to 1; and the beats of that treble with every 8th still lower are isochronous. But if the Denominator of the given base be an odd number, the beats which its treble makes with it and every 8th below it are all isochronous,

T A B. 3.

$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{4}$	1	$\frac{5}{8}$	$\frac{5}{16}$
A''	A'	A	C※	a	a'
$\frac{6}{1}$	$\frac{3}{1}$	$\frac{3}{2}$	1	$\frac{3}{4}$	$\frac{3}{8}$
A''	A'	A	E	a	a'
$\frac{20}{3}$	$\frac{10}{3}$	$\frac{5}{3}$	1	$\frac{5}{6}$	$\frac{5}{12}$
A''	A'	A	F※	a	a'

For the major term of the perfect ratio of any Major consonance is an Odd number (k), and is the Numerator of the exponent of its base (that of its treble being 1) and its denominator continually doubled gives the successive denominators

(k) Sect. 2. Art. 1. Table,

minators of the exponents of all the ascending 8^{ths}, and continually halved, if it be an even number, gives those of the descending 8^{ths}, till it be reduced to an Odd Number; which continues to be the denominator of all the exponents still lower. But if the Denominator of the given base be Odd, it is itself the denominator of the exponents of all the inferior 8^{ths}. Therefore the law of the beats is evident by Art. 1.

4. Tab. 1. 2. 3. Hence any two imperfect consonances which compose a perfect 8th, will beat equally quick, if the minor consonance be below the major; but if the minor be above the major, it will beat twice as quick as the major: the denominators of the exponents of the base and treble of the 8th being equal in the first case and as 2 to 1 in the second.

5. These examples are sufficient to shew the agreement of the theory of beats with the easiest experiments, as requiring no more to be done in many instances than to examine by the ear whether the successive 8^{ths}, as A', A, a, a', &c, throughout the scale of the organ or harpsichord be quite perfect, and if not, to make them so. For the consonances which compose those 8^{ths} being made imperfect, as they usually are and ought to be, the ear will judge very well whether the beats of such concords as by theory ought to be isochronous, are really so or not when sounded immediately after one another.

6. These experiments attentively tried will be perceptible in some degree upon a single stop of a good harpsichord, and very plainly upon the
the

the open diapason of a good organ; where the beats of the simpler concords about the middle of the scale will be very distinct and slow enough to be easily counted. The equal times of beating may be measured by a watch that shews seconds or a simple pendulum of any given length: and if the blast of the bellows be sufficiently uniform, it may be questioned whether an 8th may not be tuned perfect or nearer to perfection by the isochronous beats of a minor and major concord which compose it (*l*) than by the judgment of the most critical ear.

7. Pl. XII. Tab. I. Of consonances which have a common sound and a common temperament, the simpler generally beat slower than the less simple do; the denominators of the exponents of the simpler being generally smaller (*m*).

Scholium 3.

1. *Mersennus* and Mr. *Sauveur* are the only writers I know of that take any notice of the physical cause of the beats of consonances. *Sauveur* imagined that they beat at every coincidence of their pulses (*n*), and observing that
he

(*l*) Art. 4. of these. (*m*) Sect. III. Art. 5.

(*n*) M. *Sauveur* ayant cherché la cause de ce Phenomene, a imaginé avec une extrême vraisemblance, que le son des deux tuyaux ensemble devoit avoir plus de force, quand

he could distinguish the beats pretty well when they went no quicker than 6 in one second, and still plainer when they went slower, he concluded that he could not perceive them at all when they went quicker than at that rate (*o*); and thence he inferred that octaves and other simple concords, whose vibrations coincide very often, are agreeable and pleasant because their beats are too quick to be distinguished, be the pitch of the sounds ever so low; and on the contrary, that the more complex consonances whose vibrations coincide seldomer, are disagreeable because we can distinguish their slow beats; which displease the ear, says he, by reason of the inequality of the sound (*p*). And in pursuing this thought he found, that those consonances which beat faster than 6 times in a second, are the very same that musicians treat as concords; and that others which beat slower are the discords; and he adds, that when a consonance is a discord at a low pitch and a
concord

quand leurs vibrations, après avoir été quelque temps séparées, venoient à se réunir et s'accordoient à frapper l'oreille d'un même coup. Histoire de l'Acad. Royale des Sciences, année 1700, pag. 171. 8^{vo}.

(*o*) Donc dans tous les accords où les vibrations se rencontreront plus de 6 fois par seconde, on ne sentira point de battemens, et on les sentira au contraire avec d'autant plus de facilité que les vibrations se rencontreront moins de 6 fois par seconde. *ibid.* pag. 176.

(*p*) Les battemens ne plaisent pas à l'oreille, à cause de l'inégalité du son, et l'on peut croire avec beaucoup d'apparence, que ce qui rend les octaves si agréables, c'est qu'on n'y entend jamais de battemens. *ibid.* pag. 177.

concord at a high one, it beats sensibly at the former pitch but not at the latter (*q*).

2. As Mr. *Sauveur* appeals to numbers, let us see what evidence they produce. The tones and sevenths major and minor being discords, must beat slower than 6 times in one second by his own hypothesis. Then let them beat but 4 or 5 times, and it will follow that the major 1vth and minor 5th cannot beat above once in a second.

For the lengths of the cycles of perfect consonances to a common base, are proportional to the lesser terms of the ratios of their vibrations (*r*), which being but 8 and 9 in the former discords and 32 and 45 in the latter (*s*), shew, that the latter must beat 4 or 5 times slower than the former, that is, as slow at least as a clock that beats seconds.

But in sounding the latter discords upon an Organ, Harpsichord or Violoncello, even at a low pitch, I find their beats are so quick that I cannot count them; or rather they do not beat at all, like imperfect consonances, but only flutter,

(*q*) En suivant cette idée, on trouve que les accords dont on ne peut entendre les battemens, sont justement ceux que les musiciens traitent de consonances, et que ceux dont les battemens se font sentir, sont les dissonances; et que quand un accord est dissonance dans une certaine octave, & consonance dans une autre, c'est qu'il bat dans l'une, et qu'il ne bat pas dans l'autre. *ibid.* pag. 177.

(*r*) Sect. III. Art, 13.

(*s*) Table of perfect ratios, Sect. 2. Art. 1.

flutter, at a slower or quicker rate according to the pitch of the sounds.

The truth is, this gentleman confounds the distinction between perfect and imperfect consonances, by comparing imperfect consonances (*t*) which beat because the succession of their short cycles is periodically confused and interrupted (*u*), with perfect ones which cannot beat, because the succession of their short cycles is never confused nor interrupted.

3. The fluttering roughness abovementioned is perceivable in all other perfect consonances, in a smaller degree in proportion as their cycles are shorter and simpler, and their pitch is higher, and is of a different kind from the smoother beats and undulations of tempered consonances; because we can alter the rate of the latter by altering the temperament, but not of the former, the consonance being perfect at a given pitch: And because a judicious ear can often hear, at the same time, both the flutterings and the beats of a tempered consonance, sufficiently distinct from each other.

Scholium 4.

1. In all tempered systems the times of the single vibrations of most of the consonances are incommensurable quantities.

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In

(*t*) Memoires de l'Acad. 1701, Systême, general, Sect. XII, maniere de trouver le son fixe. pag. 473. 8vo.

(*u*) Dem. of Prop. x.

In the system of mean tones, for instance, the single vibrations of the sounds which terminate the tone are in the ratio of $\sqrt[4]{5}$ to 2, the subduplicate of 5 to 4, as the mean tone is half the III^d. Likewise the single vibrations of v^{ths} tempered by a quarter of a comma, are in the ratio of $\sqrt[4]{5}$ to 1, the subquadruplicate of 5 to 1, as the interval of the v^{ths} is a quarter of the XVIIth or 2VIII + III. For as $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{8}{80} = \frac{1}{5}$, so $V + V + V + V - c = \text{XVII}$; whence $V - \frac{1}{4}c + V - \frac{1}{4}c + V - \frac{1}{4}c + V - \frac{1}{4}c = \text{XVII}$. Lastly the ratio of the vibrations of two sounds whose interval is a quarter of a comma, is $\sqrt[4]{81}$ to $\sqrt[4]{80}$, or $\sqrt[4]{3 \times 3 \times 3 \times 3}$ to $\sqrt[4]{2 \times 2 \times 2 \times 2 \times 5}$, or 3 to $2\sqrt[4]{5}$; and consequently the ratio of the vibrations of any consonance tempered by a quarter of a comma, is also incommensurable, as being composed of the ratio of the vibrations of the perfect consonance, and the ratio of the single vibrations which terminate its temperament.

The same may be said of any perfect consonance tempered by any aliquot part or parts of a comma; whose vibrations are always incommensurable, because 81 and 80 are not equal powers of any two numbers whatever (x). We may conclude then, that in tempered systems the vibrations of most of the consonances are incommensurable.

2. Now

(*) See coroll. Prop. 2. VIII. Elem. Euclid.

2. Now if the agreeable sensation of consonances, according to the received principle in Harmonics, be the result of the frequent coincidences of their pulses, and consequently be more or less agreeable according as the coincidences are more or less frequent; all the consonances in tempered systems, whose vibrations are incommensurable, ought to be the greatest discords in nature: it being impossible for their pulses to coincide more than once in an infinite time. For as no two numbers how large soever, can express the ratio of such vibrations, so no multiple of one vibration can ever be equal to any multiple of the other. And yet experience shews that such consonances are much more agreeable than perfect discords whose pulses coincide very often.

We may approach indeed as near as we please, and certainly much nearer than the sense can distinguish, towards the exact magnitude of an incommensurable ratio, by the ratios of whole numbers; but as these will grow larger and larger without bounds, so will the time between the successive coincidences, or the length of the approximating cycle of the pulses: by which I mean the time of either of the incommensurable vibrations multiplied by the heterogeneous term of the approximating ratio.

Let any man tell us then where we may stop, and which of those cycles it is, whose repetition excites the determinate sensation of the consonance.

3. The like difficulty occurs in approaching gradually even to a commensurable ratio of the vibrations of any perfect consonance. For if either of its vibrations be pretty much altered at once, and then be made to approach by degrees to its former length, the terms of the several approximating ratios will grow larger and larger without bounds and in regular order, except when ratios occur whose terms are reducible; and the cycles of their pulses will accordingly be longer and longer and their coincidences fewer and fewer without limit, those interruptions excepted; and yet the consonance will grow better and better by regular degrees till it arrives at perfection, as is certain by experience. For instance the ratios 30 to 21, 300 to 201, 3000 to 2001, &c, approach nearer and nearer to 3 to 2, and the v^{th} s whose vibrations are in those ratios grow more and more harmonious, though the cycles of their pulses grow longer and longer to infinity.

4. It is therefore impossible to account for the phenomena of imperfect consonances upon the principle of coincidences, which indeed is applicable to none but perfect ones. Accordingly Dr. *Wallis* (y),
Mr.

(y) It hath been long since demonstrated, that there is no such thing as a just hemitone practicable in music, and the like for the division of a tone into any number of equal parts, three, four or more. For supposing the proportion of a tone or full note to be as 9 to 8, that of the half note must be as $\sqrt{9}$ to $\sqrt{8}$, that is as 3 to $\sqrt{8}$,
or

Mr. Euler (z) and others disapprove incommensurable vibrations as impracticable and inharmonious.

5. But supposing the vibrations V, v of imperfect unisons to be incommensurable, or $V : v :: \sqrt{p} : \sqrt{q}$, and x to be an indeterminate vibration, and $V : x :: m : n$, and the ratios of the indeterminate numbers m, n to approach gradually to the given ratio of \sqrt{p} to \sqrt{q} ; though the length $n V, = m x$, of the indeterminate cycle of the pulses of V and x , increases without bounds, nevertheless the length $\frac{n}{m-n} V, = \frac{m}{m-n} x$, of the indeterminate period of their pulses tends gradually to a determinate limit $\frac{\sqrt{q}}{\sqrt{p}-\sqrt{q}} V = \frac{\sqrt{p}}{\sqrt{p}-\sqrt{q}} v$. And this is the period of the pulses of the incommensurable vibrations V, v , which excites the determinate sensation of the imperfect unisons, be the complex cycle of their pulses ever so long, infinite or impossible.

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or as 3 to $2\sqrt{2}$, which are incommensurable quantities; and that of a quarter note as $\sqrt[4]{9}$ to $\sqrt[4]{8}$, which is yet more incommensurate; and the like for any other number of equal parts: which will therefore never fall in with the proportions of number to number. *Upon the imperfection of an Organ.* Phil. Trans. N^o. 242, or Abridgm. vol. I. p. 705. edit. I.

(z) Denique ob nullam sonorum rationem rationalem præter octavas, hoc genus [musicum] harmoniæ maxime contrarium est censendum; etiam si hebetiores aures discrepantiam vix percipiant. *Tentamen novæ Theoriæ musicæ*, cap. IX. sect. 17. Petropoli. 1739.

I say determinate sensation. For though the alternate lesser intervals of the pulses in the several successive periods of V and v , even when commensurate, are not precisely equal (*a*), yet it is highly probable that the ear could not distinguish a repetition of any one period from the succession of them all, and seems agreeable to experience in observing the identity of the tone of imperfect unisons held out upon an organ.

6. For further illustration I will add an example or two. We shewed above that the vibrations V , v of the mean tone are as $\sqrt{5 : 2} :: 2.23606796 \text{ \&c} : 2 :: m : n$. Whence the length of the period of the pulses of V and v , is $\frac{n}{m-n} V = \frac{2V}{0.23606796 \text{ \&c}} = 8.47213 \text{ \&c} \times V$; which is a medium between $8V$ and $9V$, the cycles of the pulses of the major and minor tones, something less than the arithmetical, or even the geometrical mean, but not quite so little as the harmonical mean between them (*b*).

Again, when V and v are the vibrations of two sounds whose interval is a quarter of a comma, we found $V : v :: 3 : 2 \sqrt[4]{5}$ or $2.99069756 \text{ \&c} :: m : n$; whence the period of the pulses of V and v is $\frac{n}{m-n} V = \frac{2.99069756 \text{ \&c}}{0.00930244 \text{ \&c}} \times V = 321.4960 \text{ \&c} \times V$.

Or

(*a*) Coroll. 2. Prop. vii.

(*b*) See Sect. vii. Def. ii.

Or thus. In approximating towards the ratio of V to v , or 3 to $2\sqrt[4]{5}$, or 3 to 2.990697, or 3000000 to 2990697 by small numbers (c), the ratios greater than V to v are 322 to 321, 967 to 964, 1612 to 1607, &c. Whence the cycle 321 V and the periods $321\frac{1}{3}V$, $321\frac{2}{5}V$, &c, are all too short.

And the ratios less than V to v being 323 to 322, 645 to 643, &c, the cycle 322 V and periods $321\frac{1}{2}V$, &c, are all too long. Therefore the true period falls between the last mentioned limits, agreeably to the former computation.

From what has been said of imperfect unisons the difficulty vanishes in other imperfect consonances, by observing the reduction of the periods of their imperfections to those of imperfect unisons, as in Prop. VIII.

7. *If the isochronous vibrations of contiguous parcels of air, excited by different strings, cannot be reduced to a synchronism by the mutual actions of the particles, (as I think they cannot,) it will follow that coincident pulses are not necessary but only accidental to a perfect consonance.*

For while an imperfect consonance is sounding, if the ratio of the vibrations be made perfect, as in tuning a musical instrument, from the instant of this change the dislocation of the pulses, whatever it be, will continue unal-

G 4 tered

(c) See Mr. Cotes's Harmonia Mensurarum, Prop. 1. Schol. 3.

tered in all the subsequent short cycles ; and thus the consonance is perfect without any coincident pulses, unless when the change of the ratio happens at the instant of the coincidence of two pulses.

8. *This however seems indisputable, that coincident pulses are not necessary to such harmony as the ear judges to be perfect.*

For if any long period of imperfect unisons, intercepted between two beats, be lengthened greatly and indeterminately, as in tuning an instrument ; any given part of it, as long as any musical note, will approach indefinitely near to perfect unisons ; certainly nearer than the ear can distinguish, as being often doubtful of their perfection. And yet throughout that part (supposed to be small in comparison to the whole period) the pulses of one sound divide the intervals of the pulses of the other very nearly in a given ratio, of any determinate quantity between infinitely great and infinitely small, in proportion to the distance of that part from the periodical point or point of coincidence. Nevertheless the ear cannot distinguish any difference in the harmony of such different parts, as is evident by often repeating the same consonance, which can hardly begin constantly in the same place of the long period. And the same argument is applicable to any given consonance, as being formed by intermitting a proper number of pulses of each sound of the imperfect unisons : and the

the conclusion seems to be confirmed by the following experiment.

9. When any string of a violin or violoncello is moved by a gentle uniform bow, while its middle point being lightly touched by the finger, is kept at rest, but not pressed to the fingerboard ; the two halves of the string will sound perfect unisons, an eighth above the sound of the whole ; and will keep moving constantly opposite ways.

Because the tension and stiffness of the parts of the string on opposite sides of the quiescent point, compel them to opposite and synchronous motions, and these parts compel the next to the like motions, and so on, to the ends of the string. Hence, because these opposite motions of the halves of the string communicate and propagate the like motions to the contiguous particles of air and these to the next successively, it follows that different particles of air at the ear, placed any where in a perpendicular that bisects the whole string, will keep moving constantly opposite ways at the same time ; those particles, which received their motion from one half of the string, going towards the ear, while others are returning from it, which received an antecedent motion from the other half of the string : Or, in fewer words, the successive pulses of one sound are constantly bisecting the intervals between the pulses of the other : And yet the harmony of the unisons is
perfectly

perfectly agreeable to the ear, as I have often experienced.

10. And in so rare a fluid as air is, where the intervals of the particles are 8 or 9 times greater than their diameters (d), there seems to be room enough for such opposite motions without impediment: especially as we see the like motions are really performed in water, which in an equal space contains 8 or 9 hundred times as many such particles as air does (d). For when it rains upon stagnating water, the circular waves propagated from different centers, appear to intersect and pass through or over each other, even in opposite directions, without any visible alteration in their circular figure, and therefore without any sensible alteration of their motions.

11. If it be objected to the experiment above, that a constant bisection of the intervals of the pulses of one of the unisons by those of the other, if true, ought to excite a sensation of a single sound an eighth higher than the unisons, and as it does not, that of consequence there is no bisection; a satisfactory answer to the objection might easily be drawn from the different duration and strength of the single pulses of different sounds at a different pitch, were it necessary to enter into that consideration.

12. But

(d) Newt. Princip. Lib. 2. Prop. 50. Schol. and Prop. 23.

12. But after all, as absolute certainty is difficult to be had in this inquiry, I chose to give the vulgar definition of a perfect consonance in Sect. III. Art. 3, as a simpler principle to build upon, and yet as fit for that purpose as a more general one would be, even supposing it were incontestable.

Scholium 5.

Having observed a very strict analogy between the undulations of audible and visible objects, I will here describe it, as an illustration of the foregoing theory of imperfect consonances.

Pl. XIII. Fig. 39. Let the points a, b, c , &c and α, β, γ , &c represent the places of two parallel rows of equidistant and parallel objects, such as pales, pallisades, &c, and let them be viewed from any large distance by an eye at any point z . In a plane passing through the eye and cutting the axes of the parallel objects at right angles in the points, a, b, c , &c, α, β, γ , &c, let lines drawn from z through α, β, γ , &c, cut the line of the other row in A, B, C , &c. Then by the similar triangles ABz and $\alpha\beta z$, BCz and $\beta\gamma z$, CDz and $\gamma\delta z$, &c, we have $AB : \alpha\beta :: (Bz : \beta z ::) BC : \beta\gamma :: (Cz : \gamma z) :: CD : \gamma\delta :: \&c$. Therefore the antecedents AB , BC , CD , &c, which are to the equal consequents $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, &c, in the same ratio,

are

are also equal to one another, and are the apparent projections of the consequents upon the line abc of the other row.

Hence supposing m and n to represent the least whole numbers in the given ratio of AB to ab , we have a line $m \times ab = n \times AB$, equal to the length of the cycle between the apparent coincidences of some of the objects in one row with some in the other; as of α and a at A , of κ and m at K , &c: and if $m-n$ be not an unit we have a shorter line $\frac{m-n}{m} ab = \frac{n}{m-n} AB = AX$ or XK , equal to the length of the apparent period of their nearest approaches towards coincidences; as on each side of the point X , according to the demonstration of the VIIth proposition.

But if the point z be so situated, that the lines AB and ab or $\alpha\beta$, or Bz and βz , or Cz and γz , &c, which are all in the same ratio, happen to be incommensurable, it will be impossible, mathematically speaking, for more than one couple of objects to appear coincident (e), and yet the periods of their apparent approaches will subsist in this case as well as in the other.

Now if the objects be white, or of any colour that reflects more light to the eye than what comes to it from the spaces between them, and their breadth be considerable as usual, the rows will appear the least luminous about the
coin-

(e) See Prop. XI. Schol. 4. Art. 2.

coincident objects and the periodical points, A, X, K , &c, where the objects of the nearer row hide the whole or some part of those behind them in the remoter row; and the rows will appear gradually more luminous towards the middle of the periods, where the objects will be seen distinct from one another if they be not too broad. And the contrary will happen if the objects in the rows be less luminous than the spaces between them.

Consequently if the spectator stands still and moves his eye from one end of the rows to the other, he will see an alternate succession of light and shade; and while he moves forwards in any transverse direction $\propto \omega$, and fixes his eye upon a given place of the rows, he will then see an undulation of light and shade, moving forwards quicker or slower according to the celerity of his own motion.

For then the apparent coincidences which were at A, K , &c, and consequently the intermediate periodical points X, Y , &c, will gradually shift from A to B , &c, and from K to L , &c, as is evident from the angular motion of the visual rays about the fixt points or objects α, β, γ , &c, κ, λ, μ , &c: And this is a known phenomenon.

If the spectator recedes from the rows, the period $\frac{m}{m-n} ab$ will grow longer, and upon his moving transversely, the visible undulations will be broader and slower than before, and at a
very

very great distance from the rows, will become imperceptible; as being changed into an uniform appearance of both rows in the place of one: quite analogous to the audible undulations of imperfect unisons, as they grow slower and less perceptible while the unisons are approaching to perfection.

The like phenomenon results from two rows of pales that meet in any angle.

PROPOSITION XII.

Imperfect consonances of the same Name are equally harmonious when their short cycles are equally numerous in the periods of their imperfections.

As perfect consonances of the same Name are equally harmonious because their cycles are similarly divided by the pulses of their sounds; so imperfect consonances will be equally harmonious when their periods are similarly divided.

Hence all imperfect unisons whose single vibrations have the same ratio, are equally harmonious, as having similar periods (f); and therefore all imperfect consonances of the same name whose tempering ratios are the same, are equally harmonious.

For

(f) Prop. VII. Cor. 4.

For since the vibrations of the corresponding perfect consonances have the same given ratio, m to n , and the vibrations of the imperfect ones are derived from those of the similar unisons by intermitting $m-1$ and $n-1$ pulses of their homologous vibrations, so as to leave equidistant pulses in every series (g); the similar periods of the unisons are thereby altered into similar periods of imperfect consonances; and the equal intervals of the unisons into equal temperaments of the consonances (b).

And the lengths of these similar periods being proportional to the single vibrations of their bases or to equimultiples of them, that is, to the lengths of the short cycles of the perfect consonances, will contain equal numbers of imperfect short cycles (i). Q. E. D.

Coroll. Consonances of the same name are equally harmonious when equally and similarly tempered.

Scholium.

After an organ had been well tuned by making all the tempered v^{ths} as equally harmonious as the ear could determine, I found that the numbers of their beats, made in equal times, were inversely proportional to the times of

(g) See Dem. Prop. VIII. towards the end “ And “ Universally, &c.

(b) Prop. VIII. Cor. 1, 2.

(i) Ibid. Cor. 2.

of the single vibrations of their bases or trebles, as nearly as could be expected: or that the times between their successive beats, which are equal to the periods of their least imperfections (k), were directly proportional to those homologous vibrations, or to equimultiples of them, or to the lengths of the short cycles, which therefore were equally numerous in those periods.

PROPOSITION XIII.

Imperfect consonances of all sorts are equally harmonious, in their kind, when their short cycles are equally numerous in the periods of their imperfections.

Pl. XII. Fig. 34. The times of the single vibrations of imperfect unisons being represented by AB and ab , let AD and ac , that is $3AB$ and $2ab$ be those of imperfect v^{ths} . And one length of their imperfect short cycle being $2AD = AG$, and the other being $3ac = ag$, their difference Gg is the dislocation of the pulses G, g at the end of the first short cycle $AagG$, measured from the coincident pulses Aa . And the greater of the two dislocations which terminate the several succeeding cycles, is double, triple, &c of Gg (l).

Again,

(k) Pro. x. (l) Prop. vii.

Again, conceiving the pulses c, g, l , &c, to be now intermitted, let AD and ae , that is $3AB$ and $4ab$ be the single vibrations of imperfect 4^{th} s. And the two lengths of their first short cycle $ANna$ being $4AD=AN$ and $3ae=an$, their difference Nn is the dislocation of the pulses N, n at the end of that cycle; and in the several succeeding cycles the greater of the two dislocations is double, triple, &c of Nn .

And the common period AZ or an of those dislocations or imperfections in the short cycles of the v^{th} s and 4^{th} s, is the same as the period or simple cycle of the pulses of the vibrations AB, ab of the imperfect unisons (m).

Now the two dislocations Gg, Nn , in the first imperfect cycles of the v^{th} s and 4^{th} s in that period, are in the ratio of AG to AN (n), the lengths of the cycles, that is of $2AD$ to $4AD$, or 1 to 2: and the two greater dislocations Xy, Qr , in the last imperfect cycles $Xy \varepsilon \Delta$, $Qr \varepsilon \Delta$, in the same period AZ , are in the ratio of their distances ZX, ZQ , from this end of it: and this ratio is less than that of ΔX to ΔQ , or 1 to 2. But the two greater dislocations $K\lambda, \Pi\sigma$ in the subsequent cycles $K\lambda \varepsilon \Delta$, $\Pi\sigma \varepsilon \Delta$, of the next period, are in the ratio of ZK to $Z\Pi$, which, on the contrary, is greater than that of ΔK to $\Delta \Pi$, or 1 to 2.

H

The

(m) Prop. VIII. (n) Prop. VII.

The periods must be conceived to contain a much greater number of short cycles than can be well represented in a scheme. And then, as the corresponding dislocations in the v^{th} and 4^{th} lie farther and farther from Z , the ratio of their distances and magnitudes will approach nearer and nearer to 1 to 2.

Therefore 1 to 2, or the ratio of the lengths of the short cycles of the v^{th} and 4^{th} , is either the exact or the mean ratio both of the greater and the lesser dislocations in all their corresponding short cycles: because the lesser of the increasing dislocations in any subsequent cycle, is the same as the greater in the antecedent one.

Now while the length AG or ag remains unaltered, imagine the dislocation Gg of the v^{th} to be increased in that ratio of 1 to 2, and then it will be equal to the former magnitude of the dislocation Nn of the 4^{th} , or to Nn in Fig. 35, supposing the pulses C, G, L , &c to be absent. And the first dislocation Bb of the pulses B, b , of the imperfect unisons, being at the same time increased in the same ratio, their period AZ , which is also that of the dislocations in the $v^{\text{th}}(o)$, will be diminished very nearly in that ratio inverted (p). And thus the present period of the imperfect v^{th} and the former period of the 4^{th} , are in the ratio of the lengths of their short cycles; which therefore

fore are equally numerous in their respective periods.

And since the greater and lesser dislocations at the ends of the corresponding subsequent short cycles of the v^{th} and 4^{th} , are now respectively equal, either exactly or at a medium of one with another, and equally numerous too, the whole periods composed of these short cycles, will be equally harmonious. Because those equal dislocations of the pulses in the corresponding short cycles, are the causes that spoil their harmony: and causes constantly equal will have equal effects.

The conclusion will be the same if the dislocation Nn , in the first cycle of the 4^{th} in either figure, be contracted to the magnitude of the dislocation Gg belonging to the v^{th} in the other. For then the new period of the 4^{th} , being double of the old one (q), will be to the old one, or that of the v^{th} , as AN to AG , that is, in the ratio of the lengths of their short cycles, which therefore are equally numerous in these periods: and the dislocations at the ends of the several subsequent short cycles of the 4^{th} , being likewise contracted to the respective magnitudes of those of the v^{th} , the consonances are again made equally harmonious.

And lastly, since either of those consonances is equally harmonious to another of the same name, at any other pitch, when their short

H 2

cycles

cycles are equally numerous in their periods (r), it appears that 4^{th} and v^{th} are equally harmonious at any pitches, when their short cycles are equally numerous in their periods. And the like proof is plainly applicable to any other case of these or any other consonances: I mean when the common period of the imperfect unisons is terminated at first either by coincident pulses or periodical points; as will plainly appear by conceiving a short cycle or two to result from a proper intermission of the pulses of imperfect unisons on each side of such points in fig. 24, 25. Q. E. D.

Coroll. 1. Imperfect consonances are more harmonious in the same order as their short cycles are more numerous in the periods of their imperfections.

For if any two imperfect consonances be supposed equally harmonious, their short cycles will be equally numerous in their periods, by the proposition. Then if either of the given periods be lengthened, the short cycles will be more numerous in it, and the least dislocation of their pulses being smaller than before, and the greatest much the same (s), the dislocations will first increase and then decrease by smaller and more degrees from one end of the period to the other. And thus the consonance will be more harmonious than it was at first, or than the other given consonance.

And

(r) Prop. XII. (s) Prop. VII and VIII.

And on the contrary, if the period of either consonance be shortened, the number of its short cycles will be diminished, and the dislocations of their pulses will increase and decrease by larger and fewer degrees than before. And thus the consonance will be less harmonious than it was before, or than the other given consonance.

Coroll. 2. Imperfect consonances are more harmonious in the same order, as their temperaments multiplied by both the terms of the ratios of the single vibrations of the corresponding perfect consonances, are smaller; and are equally harmonious when those products are equal.

Pl. XII. Fig. 34, 35. For the vibrations of imperfect unisons being AB and ab and the terms of any perfect ratio of majority m and n , the vibrations of an imperfect consonance tempered sharp are mAB and nab , and those of the imperfect consonance tempered flat are mab and nAB ; and the periods of the least imperfections in both have the same length as the period of the imperfect unisons (t); which length, supposing $AB : ab :: R : r$ in the least integers, is $\frac{r}{R-r} AB$; call it p .

Now the length of the imperfect short cycle of either of those imperfect consonances is $mnAB$ (u); call it c . Then

$$\begin{array}{r} H \\ 3 \end{array} \qquad \frac{p}{c}$$

(t) Prop. VIII. (u) Prop. VIII. Cor. 2.

$\frac{p}{c} = \frac{r}{R-r} AB \times \frac{1}{mnAB} = \frac{1}{mn} \times \frac{r}{R-r} = \frac{1}{mnt}$ by taking $t = \frac{R-r}{r}$, which being as the logarithm of the tempering ratio $R : r$, or AB to ab (x) is very nearly as the temperament of both those consonances (y).

Therefore in the same order in which the values of $\frac{p}{c}$ or $\frac{1}{mnt}$ are greater, or the values of mnt are smaller, the corresponding consonances are more harmonious, by corol. 1; and are equally harmonious when the values of mnt are equal, by the present proposition.

Coroll. 3. Consequently imperfect consonances are equally harmonious when their temperaments have the inverse ratio of the products of the terms of the perfect ratios of the corresponding perfect consonances.

For when the values of the product $mn \times t$ are equal, the values of t have the inverse ratio of the values of mn .

Coroll. 4. When the products mn of the terms of the perfect ratios are equal, the tempered consonances are more harmonious in the same order as their temperaments are smaller; and are equally harmonious if their temperaments be equal.

For

(x) Cor. 2. Lemma to Prop. IX and Prop. VIII cor. 1.

(y) Sect. I. Art. 11.

For if the values of $m n$ or $\frac{1}{m n}$ be equal, the values of $\frac{p}{c} = \frac{1}{m n} \times \frac{1}{t}$ are greater in the same order as those of $\frac{1}{t}$ are greater, or as those of t are smaller; and are equal when the values of t are equal.

Coroll. 5. Therefore imperfect consonances of the same Name are more harmonious in the same order as their temperaments are smaller; and are equally harmonious when they are equal.

Because the terms of the perfect ratios of consonances of the same name are the same, and their product the same.

Coroll. 6. Imperfect consonances equally tempered are more harmonious in the same order as the products of the terms of the perfect ratios belonging to the perfect consonances are smaller; and are equally harmonious when those products are equal.

For the values of t being supposed equal, those of $\frac{p}{c} = \frac{1}{m n} \times \frac{1}{t}$ are greater in the same order as the values of $\frac{1}{m n}$ are greater, or as those of $m n$ are smaller; and the former values are equal when the latter are so.

Coroll. 7. Imperfect consonances equally tempered are generally more harmonious in the same order as they are simpler, the pure ones chiefly excepted (z), which are more harmo-

H 4

nious

(z) Sect. III. Art. 8.

nious than some others that are simpler; though separately considered they follow that order exactly.

This will appear from the sixth corollary by a series of the products of the terms of the ratios in the first column, compared with the series of numbers in the second column of the table in Sect. III. Art. 5, shewing the order of the simplicity of consonances.

Coroll. 8. Consequently simpler consonances will generally bear greater temperaments than the less simple will; or the less simple ones generally speaking will not bear so great temperaments as the simpler will: contrary to the common opinion (*b*).

Coroll. 9. The tempered concords in the system of mean tones (*c*) are not equally harmonious in their kinds.

For by Coroll. 6, and by inspection of the terms of the perfect ratios annexed to the characters of the concords in the first of the tables
in

(*b*) Octavæ autem fiant exactæ; nam vel minimus octavæ defectus fit intolerabilis. *Dechales* Cursus math. Tom. IV. de Musica, cap. XI.

(*b*) Octavarum autem omnium unica est species, eaque perfecta ratione 1 ad 2 contenta. Hoc enim intervallum, propter perfectionem, vix aberrationem à ratione 1 ad 2 pati posset, quin simul auditus ingenti molestia afficeretur. Namque quo perfectius perceptuque facilius est intervallum, eo magis sensibilis fit error minimus; minus autem sentitur exigua aberratio in intervallis minus perfectis. *Tentamen novæ Theoriæ musicæ*, cap. IX. sect. 10. *Petropoli* 1739.

(*c*) Prop. 2.

in the next section, it will appear, that the v^{th} and 4^{th} and their compounds with $viii^{\text{ths}}$, are more harmonious than the vi^{th} and 3^{d} and their compounds with equal numbers of $viii^{\text{ths}}$, as being all equally tempered in that system (*d*).

Coroll. 10. The harmony of those concords is still more unequal in the *Hugenian* system, resulting from a division of the octave into 31 equal intervals (*e*).

Because the common temperament of the vi^{th} and 3^{d} and their compounds with $viii^{\text{ths}}$, which by *Coroll. 4* and 9, should be smaller than that of the v^{th} and 4^{th} and their compounds with $viii^{\text{ths}}$, to render them equally harmonious, is on the contrary something greater.

Coroll. 11. Imperfect consonances are more harmonious both as they beat slower, and as the cycles of the perfect consonances are shorter.

For the quantities $\frac{p}{c}$ will be greater on both accounts (*f*) and the harmony better (*g*).

Coroll. 12. Imperfect consonances having the same Base are more harmonious in the same order as their Beats made in equal times and multiplied by the Minor terms of the perfect ratios of the respective perfect consonances are smaller: and are equally harmonious when those products are equal, that is, when the beats are inversely as the minor terms (*b*).

For

(*d*) Prop. III. Coroll. 3.

(*e*) See Prop. XVII, Scholium.

(*f*) Prop. XI.

(*g*) Prop. XII and XIII.

For let the single vibrations of the base and treble of an indeterminate perfect consonance be Z and z , and $Z : z :: m : n$ in the least numbers, then the short cycle $c = nZ = mz$, and putting β for the number of beats made in any given time by the corresponding imperfect consonance, the period p is as $\frac{1}{\beta}$ as being equal to the interval of the successive beats (i); and the harmony being as $\frac{p}{c}$ or $\frac{1}{\beta nZ}$ or $\frac{1}{\beta mz}$, is better as the values of βn are smaller if Z be constant, or as βm is smaller, if z be constant, by Coroll. 1. Prop. XII.

Coroll. 13. Hence if the Bases and Beats be the same, the harmony is better as the minor terms are smaller and equally good when they are the same: or if the Bases and Minor terms be the same, it is better as the beats are slower, and equally good when they are isochronous.

Coroll. 14. And the two last corollaries are applicable to trebles and major term, by reading trebles instead of bases and major terms instead of minor, as appears by the demonstration.

(*b*) See Pl. I. Fig. 3. and Plate XII. Tab. I.

(*i*) Prop. x.

SECTION

SECTION VII.

Of a system of sounds wherein as many concords as possible, at a medium of one with another, shall be equally and the most harmonious.

DEFINITION I.

The Arithmetical Mean among any number of quantities, is to the sum of them under their given signs, as an unit is to their number; and has the same sign as their sum has: Or if they be expressed by numbers, it is the quotient of their sum divided by their number.

Thus the arithmetical mean among the quantities $a, b, c, -d$, is $\frac{a+b+c-d}{4}$.

Coroll. 1. Hence the sum of the excesses of all the greater quantities above their arithmetical mean, is equal to the sum of the defects of all the smaller from the same.

For let the arithmetical mean $\frac{a+b+c-d}{4} = r$, then $a+b+c-d = 4r = r+r+r+r$.
Whence

Whence if a and b be severally greater than r , we have $a-r+b-r=r-c+r+d$.

Pl. XIII. Fig. 40. Accordingly if the lines ao , bo , $co-do$ and ro , be proportional to $a, b, c, -d$ and r , the sum of the parts ra , rb on one side of the point r , is equal to the sum of the parts rc , rd on the other side of it.

Coroll. 2. If any quantity q be added to, or taken from every one of the quantities $a, b, c, -d$, their arithmetical mean will accordingly be augmented or diminished by that quantity q .

For let $a+b+c-d=4r$, then r is their arithmetical mean. But $a+q+b+q+c+q-d+q=4r+4q=4\times r+q$, and therefore $r+q$ is the arithmetical mean among those augmented quantities $a+q$, $b+q$, $c+q$, $-d+q$: and by changing the sign of every q , it appears that $r-q$ is the like mean among the diminished quantities $a-q$, $b-q$, $c-q$, $-d-q$.

Coroll. 3. If every one of the quantities $a, b, c, -d$, be increased or diminished in any given ratio of 1 to n , their arithmetical mean will also be increased or diminished in the same given ratio.

For let $a+b+c-d=4r$, then r is their arithmetical mean. But $na+nb+nc-nd=4nr$, and therefore nr is the arithmetical mean among the quantities na , nb , nc , $-nd$.

DEFINITION II.

The Harmonical Mean among any number of quantities, is the reciprocal of the arithmetical mean among their reciprocals.

For instance, the reciprocals of a, b, c , are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, whose arithmetical mean is $\frac{1}{3} \times \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

and its reciprocal $\frac{1}{\frac{1}{3} \times \frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$ is the harmonical mean among a, b, c ; where 1 signifies any given constant quantity.

Pl. XIII. Fig. 41. Likewise in any hyperbola where the ordinates parallel to an asymptote are the reciprocals of their abscisses, measured from the center upon the other asymptote; if an absciss ro be the arithmetical mean among the abscisses ao, bo, co, do , its ordinate re is the harmonical mean among the ordinates $a\alpha, b\beta, c\gamma, d\delta$ by the definition, re being the reciprocal of the arithmetical mean ro among their reciprocals, ao, bo, co, do .

Likewise if an ordinate $m\mu$ be the arithmetical mean among the ordinates $a\alpha, b\beta, c\gamma, d\delta$, its absciss mo is the harmonical mean among their abscisses, ao, bo, co, do : and on the contrary.

Coroll.

Coroll. 1. The arithmetical mean is greater than the harmonical mean among the same quantities, if they all have the same sign.

For let the line $\beta\varrho$, produced through the top of either of the ordinates next to $r\varrho$, cut the rest in f, g, h , and the asymptote ro in e . Then because ro is the arithmetical mean among ao, bo, co, do , the line re is the arithmetical mean among the lines ae, be, ce, de (k); and $r\varrho$ the arithmetical mean among the proportional lines $af, b\beta, cg, dh$ (l), which, excepting the common ordinate $b\beta$, are severally smaller than the hyperbolic ordinates, $a\alpha, b\beta, c\gamma, d\delta$; whose arithmetical mean $m\mu$ is therefore greater than $r\varrho$ (m), the harmonical mean among the same ordinates.

Coroll. 2. The difference between the arithmetical and the harmonical means among the same quantities, will be very small when the differences of the quantities themselves are so.

This will appear by conceiving the ordinates $a\alpha, b\beta, c\gamma, d\delta$ to approach gradually towards one another till they coincide. For then the differences between the hyperbolic ordinates $a\alpha, b\beta, c\gamma, d\delta$, and the lines $af, b\beta, cg, dh$, and consequently between their arithmetical means $m\mu, r\varrho$, will gradually decrease to nothing. But $r\varrho$ is also the harmonical mean among those ordinates.

Coroll.

(k) Defin. 1. Coroll 1 or 2.

(l) Defin. 1. Coroll. 3.

(m) Defin. 1.

Coroll. 3. By increasfing every quantity in any given ratio, the harmonical mean among them will be increafed in the fame ratio.

For the reciprocals of the increafed quantities, and the arithmetical mean among them (n), will feverally be diminifhed in that ratio; and the reciprocal of this mean, which is the harmonical mean among the increafed quantities, will of confequence be increafed in the fame ratio.

Coroll. 4. Fig. 42, 43. Whatever be the figns of the propofed quantities $a\alpha$, $b\beta$, $c\gamma$, $d\delta$, their harmonical mean $r\rho$ has always the fign of the fum of their reciprocals $a\phi$, $b\phi$, $c\phi$, $d\phi$, or of $r\phi$, the arithmetical mean among them.

For the reciprocal of each quantity has the fign of the quantity itfelf, and according as their fum is affirmative or negative, fo is their arithmetical mean (ϕ), and fo is its reciprocal, or the harmonical mean among the propofed quantities.

(n) Defin. I. Coroll. 3. (ϕ) Defin. I.

P R O-

PROPOSITION XIV.

Instead of several imperfect concords differently tempered and belonging to the same perfect one, if it be necessary to use but one, let the period of its imperfections be the arithmetical mean among all the periods of those concords, and it will best answer the several purposes of every one.

Because the excesses of the longer periods above the arithmetical mean are equal, one with another, to the defects of the shorter from the same, and because the arithmetical mean period is longer (*p*) and therefore more harmonious (*q*) than the harmonical mean among the same. Q. E. D.

(*p*) Def. 2. Coroll. 1. Sect. VII.

(*q*) Prop. XIII. Coroll. 1.

P R O-

PROPOSITION XV.

The tempered concords in any one of the parcels derived from the v^{th} , vi^{th} or iii^{d} (r), whatever be their common temperament, are constantly more harmonious in the same immutable order as the products of the terms of the perfect ratios belonging to the respective perfect concords are smaller; and those concords only are equally harmonious which have equal products belonging to them; and no others can be made so, because they cannot have different temperaments while the octaves are perfect.

The truth of this proposition appears from prop. XIII. coroll. 6. and prop. III. Q. E. D.

(r) In the scholium to prop. III the concords were distributed into three parcels, which may be seen in one view in Table I placed after schol. 2. prop. XVI.

PROPOSITION XVI.

To find the temperament of a system of sounds of a given extent, wherein as many concords as possible, at a medium of one with another, shall be equally and the most harmonious.

Into the second and third columns of the 11^d Table following (*s*), transfer every couple of concords in the given system, whose characters can be taken from the different parcels in the 1st Table; omitting all other couples whose characters are both situated in any one of the parcels. And after each couple place the ratio of the temperaments which would make the two concords equally harmonious in their kind (*t*).

Then will the corollaries to the ivth, vth and vith propositions give the temperaments themselves, or the positions of the temperers *Orst*, *Orst*, &c, in Fig. 44, Pl. xiv, belonging to every one of those ratios.

Among the temperaments in the three several parcels *Gr*, *Gr*, &c, *As*, *As*, &c, *Et*, *Et*, &c, taking three harmonical means, *GD*, *AH*, *EM*, and transferring them to Fig. 45, draw three temperers *ODef*, *OgHi*, *OkIM*, and
taking

(*s*) After Schol. 2. of this prop.

(*t*) Prop. XIII. coroll. 3.

taking Eq the arithmetical mean among the three temperaments EM , Ef , Ei , the temperer $Onpq$ will approach very near to the position required in the proposition.

If greater exactness be desired, among the three temperaments in the three several parcels GD , Gg , Gk ; AH , Ae , Al ; EM , Ef , Ei , taking three harmonical means, GD' , AH' , EM' , and transferring them to Fig. 46, draw three new temperers $OD'ef'$, $Og'H'i'$, $Ok'l'M'$; and taking Eq' the arithmetical mean among the three temperaments EM' , Ef' , Ei' , the temperer $Gn'p'q'$ will approach still nearer towards the required position.

And by repeating the like construction we may approach as near as we please (u). Q.E.I.

THE DEMONSTRATION.

In combining the concords all the couples whose characters are both in any one of the parcels in Tab. I are omitted; their harmony with respect to one another, or the proportions of their periods being immutable, by reason of their common temperament (x).

Pl. XIV. Fig. 44. Now supposing Ga to be the arithmetical mean among all the temperaments

I 2

Gr ,

(u) Want of room in such small plates made it necessary to alter the true proportions of the lines in the figures; otherwise some parts of them would have appeared confused.

(x) Prop. XV.

Gr, Gr, &c, of the first parcel of concords, and drawing the temperer *Oabc*, the temperaments *Ab* and *Ec* will be the like Means among *As, As*, &c, and *Et, Et*, &c, (*y*). Whence if the periods of concords of the same name to the same base were proportional to their temperaments, the best temperer would be *Oabc* (*z*):

Because the period of the v^{th} , for instance, belonging to the temperament *Ga*, would then be the arithmetical mean among all the other periods of the v^{th} s to the same base, answering to the several temperaments *Gr, Gr*, &c. And the like may be said of the periods of the 4^{th} and of every other concord in this first parcel, as having the temperaments *Gr, Gr*, &c, common to them all, and likewise of the periods of the several concords in the other two parcels, with respect to their temperaments *Ab, As*, &c, and *Ec, Et*, &c.

But since the periods of concords of the same name to the same base are (not directly but) inversely proportional to their temperaments (*a*); the period of the v^{th} , or any other concord, belonging to the arithmetical mean temperament *Ga* is (not the arithmetical but) the harmonical mean among the other periods of that name, answering to the temperaments *Gr, Gr*, &c; and consequently is shorter (*b*) and therefore

(*y*) Def. 1. coroll. 1. 2. 3. Sect. VII.

(*z*) Prop. XIV.

(*a*) Prop. IX. coroll. 4.

(*b*) Def. 2. coroll. 1. sect. VII.

fore less harmonious than the arithmetical mean period among them (*c*) answering to the harmonical mean temperament *GD*: And what has been said of the periods in that parcel is applicable to those of the other two, with respect to the arithmetical and harmonical mean temperaments *Ab* and *AH*, *Ec* and *EM*.

Therefore the arithmetical mean periods belonging to the harmonical mean temperaments *GD*, *AH*, *EM*, would best answer the design of the proposition, if the points *D*, *H*, *M* were all situated in one temperer.

For since the sums of the temperaments terminated at the several temperers *Orst*, *Orst*, &c, are the least that can render the concords in each couple equally harmonious in their kind (*d*), it follows that the sums of all the temperaments *Gr*, *Gr*, &c, in the first parcel, of all the temperaments *As*, *As*, &c, in the second, and of all the temperaments *Et*, *Et*, &c, in the third, taking one sum with another, are also the least possible: the sum total being the same in both distributions of the particulars.

The sum of the harmonical temperaments *GD*, *AH*, *EM* being therefore the least possible (*e*), and that of all the corresponding arithmetical mean periods being the greatest (*f*), would render the system of periods at a me-

I 3 dium

(*c*) Prop. XIII. coroll. I.

(*d*) Prop. IV. v. VI.

(*e*) Def. I, and cor. I. Def. 2. sect. VII.

(*f*) Prop. IX. cor. 4.

dium of one with another, the most harmonious.

But in reality the three harmonical points D , H , M cannot fall into any one temperer. For the concords in the first parcel being simpler than those in the second and third (g) and therefore requiring a smaller temperament (b), it appears, by cor. 6, 7, 8, prop. III, that the best temperer of the system must lie within the angle AOE , and so must the arithmetical mean temperer $Oabc$, as lying not far from the best; and therefore must have the points G , E on one side of it and A on the other: And the harmonical means GD , EM , AH being less than the respective arithmetical means Ga , Ec , Ab (i), the points D and M must lie on the same side of $Oabc$ as G and E do, and H on the other side. Therefore if a temperer could pass thro' D and M , yet it could not pass thro' H .

Pl. xv. Fig. 45. In the solution of the problem it was therefore necessary to reduce the three temperers $ODef$, $OgHi$, $OkLM$ to one, by so drawing the temperer $Onpq$, as to make Eq the arithmetical mean among EM , Ef , Ei , and consequently Ap the like mean among AH , Ae , Al , and Gn the like mean among GD , Gg , Gk (k).

Now the differences of the three temperaments in each of those parcels being but small,
as

(g) Tab. I following, compared with art. 5. sect. III.

(b) Prop. XIII. cor. 8.

(i) Defin. 2. cor. I. sect. VII.

(k) Def. I. coroll. I. 2. 3.

as will appear by the following calculation (*l*), the arithmetical means among them will differ but little from the respective harmonical means among the same (*m*), which would be fitter for the purpose if their extremities *D'*, *H'*, *M'* could be situated in any one temperer (*n*). Consequently as the temperer *Onpq* falls in the middle among the three temperers conceived to pass through the harmonical points *D'*, *H'*, *M'*, it will nearly answer the several purposes of those three, and approach very near to the situation of the required temperer.

Pl. XVI. Fig. 46. Hence and by prop. XIV, it appears that a repetition of this last construction, as described in the solution, will give a temperer *On'p'q'* approaching still nearer to the required situation. Because the latter temperaments *EM'*, *Ef'*, *Ei'* differ less from one another (*o*) and consequently from their arithmetical mean *Eq'*, than the former, *EM*, *Ef*, *Ei*, did from one another and from their arithmetical mean *Eq*.

And as the same may be said of the temperaments of the other two parcels, it appears that by a further repetition of the same construction, we may find a temperer approaching as near as we please towards the position required in the proposition. Q. E. D.

Coroll. I. Fig. 45. The comma, or four times the line *GI*, being the unit, and supposing any

I 4
three

(*l*) Tab. VI. column 2.

(*m*) Def. 2. coroll. 2. (*n*) Prop. XIV.

(*o*) Tab. VII. at the end of it.

three temperaments of different parcels to be given, as $GD=d$, $AH=b$ and $EM=m$, it will be easy to collect, (from the similar triangles under the line $O13E$, the three temperers $ODef$, $OgHi$, $OkLM$, and the three parallels GD , AH , EM), that $Gg = \frac{1-b}{3}$ and $Gk = \frac{1+m}{4}$; $Ae = 1-3d$ and $Al = \frac{1-3m}{4}$; $Ef = 4d-1$ and $Ei = \frac{1-4b}{3}$; provided the three temperers be all situated within the angle EOA ; but if OH or OM lies out of it beyond A or E respectively, the sign of b or m will accordingly be changed in those theorems.

Coroll. 2. Hence we have the three arithmetical mean temperaments, $Gn = \frac{1}{3} \times \overline{d + \frac{1-b}{3} + \frac{1+m}{4}}$, $Ap = \frac{1}{3} \times \overline{1-3d+b+\frac{1-3m}{4}}$, and $Eq = \frac{1}{3} \times \overline{4d-1+\frac{1-4b}{3}+m}$.

Scholium I.

Pl. xvii. Fig. 47 serves to illustrate part of the demonstration of the proposition, by representing to the eye the proportions of the periods of the concords. It is thus constructed. The line AI being parallel to EO , the middlemost parcel of hyperbolas $\propto \frac{1}{5}$, $\propto \frac{1}{6}$, $\propto \frac{1}{10}$, $\propto \frac{1}{12}$, &c, are drawn to the asymptotes AI , $A3$; and their ordinates to their common absciss $A3$ are made proportional to the fractions $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{12}$, &c.

Hence

Hence when the imperfect VI, 3^d , VI + VIII, 3^d + VIII, &c, to the same base are each tempered by a quarter of a comma, represented by the common absciss A_3 , their periods are proportional to those ordinates (p); and when they have any other temperament represented by the absciss A_s , their periods are then proportional to the ordinates sv , sx , sy , sz , &c. (q).

And the like construction being made for the other two parcels of concords, the ordinates erected from the interfections r , s , t of any temperer $Orst$, shew the proportions of the periods in the whole system: and these proportions are the same whatever be the unit of the fractional ordinates.

Scholium 2.

In order to calculate the required temperaments of a system of any given extent, it will not be amiss to explain the following tables.

1. According to the solution of the problem, see whether every two characters of the concords, each of which lie in the different parcels in Tab. I, be placed over against one another in the second and third columns of Tab. II^d. Part I.

2. Then examine whether the ratios placed after those characters in the 4th column of that table, be rightly deduced from the fractions annexed

(p) Prop. IX. coroll. 5. and Tab. I. at the end of the next Scholium.

(q) Prop. IX. coroll. 4.

nexed to the same characters in Tab. I, according to the rule in prop. XIII. coroll. 3.

3. See whether all the temperaments in Tab. IV be rightly deduced from those ratios in Tab. II^d, by the corollaries to prop. IV, V, VI; and whether the numbers in the first column of each table correspond to the same ratios and concords.

4. Examine whether the reciprocals in Tab. V, of the temperaments in Tab. IV be right, that is, whether the product of the quotient by the divisor, differs from the dividend by less than half the divisor. When a reciprocal is negative, as coming from a negative temperament of the III^d or VIth, which lies wholly out of the angle *AOE*, I subtract it from 0 and place the remainder in the table instead of the reciprocal itself. Thus at N^o 10, $-6) 26 (= -4.33333 \text{ \&c.}$, which subtracted from 0 gives $\bar{5}.66667$ to be transferred to Tab. II^d Part II^d and there added to the positive reciprocals, for the sake of uniformity in the work; the integer $\bar{5}$ being only negative and the decimals .66667 affirmative. For n being any given integer, the number $n - 4 - \frac{1}{3} = n - 5 + \frac{2}{3}$.

5. See whether the reciprocals in Tab. V be rightly transferred into the respective columns of Tab. II^d Part II^d, which is readily done by means of the corresponding numbers in the first column of each table.

6. Cast up the several dozens of reciprocals in Tab. II^d Part II^d, and transfer the sums to Tab. III and there cast them up.

7. Tab.

7. Tab. VI is thus deduced from Tab. III. By the solution of the problem the fraction $\frac{12}{42.72013}$, = GD in Fig. 44, is the harmonical mean among the temperaments $Gr, Gr, \&c$; because its reciprocal $\frac{42.72013}{12}$ is the arithmetical mean among their reciprocals, as being their sum divided by their number. The same is to be understood of all the other fractions: and as the value of the temperament Eq , computed from coroll. 2. prop. XVI, comes out affirmative, by the coroll. to prop. IV, V, VI, it is part of the interval EC of the perfect III^d, and therefore is a negative temperament of that concord, or an affirmative one of its complement to the octave. This is the first approximation towards the required temperament.

8. Tab. VII contains the calculation of Eq' , the second approximation towards the true temperament of the III^d, in a system whose extent is but one octave, and is sufficiently evident from cor. 1, 2, prop. XVI, and Tab. VI. And by a like calculation the values of Eq' , in a system of two and of three octaves, will be found as put down under those of Eq in Tab. VI; care being taken in the operations to continue the quotients in decimals as far as they are just.

9. Therefore the result of the whole is this. As all the parts of musical compositions in any given place (setting aside double bases) are generally contained within three octaves, and as their harmony is stronger and better within that compass

compass than it would be in a larger; I chuse to make all the concords within every three octaves equally harmonious and no more, be the extent of the system ever so great; and consequently to diminish the III^d by $\frac{1}{9}$ comma, this being very nearly the value of the last $E q' = 0.11024$ in Tab. VI.

10. Hence in the system of equal harmony the temperaments of the v^{th} , $v\text{I}^{\text{th}}$ and III^d are $-\frac{5}{18}$, $+\frac{3}{18}$ and $-\frac{2}{18}$ of a comma respectively (*r*) and are proportional to the musical primes 5, 3 and 2. (*s*)

11. In determining these temperaments of the diatonic system, I have regarded no more consonances than the concords. 1. Because the discords are seldomer used than the concords. 2. Because the ear is generally less critical in the discords than in the concords. 3. Because a mean temperament among those of the concords and discords too, would differ from that of the concords alone, and therefore be less suitable to them.

12. Lastly I have kept the octave perfect. 1. Because it is the simplest and most harmonious

(*r*) Prop. III. and its 2^d and 3^d coroll.

(*s*) But if any one chuses to have all the concords in 4 octaves made equally harmonious, he will find by continuing the tables, that the III^d must be diminished by $\frac{987}{10000}$ of a comma, which being nearly $\frac{1}{10}$ comma, the temperaments of the v^{th} , $v\text{I}^{\text{th}}$ and III^d will then be $-\frac{11}{40}$, $+\frac{7}{40}$ and $-\frac{4}{40}$ of a comma respectively.

nious of all the concords, both in itself and its multiples. 2. Because some one interval must be kept perfect, in order to determine the variations of the temperaments of the rest (*t*). 3. Because upon several trials of keeping other intervals perfect instead of the octave, many reasons have occurred to me for rejecting every one of them.

13. Does it not follow then, that the system of equal harmony, as above derived from the best system of perfect intervals (*u*), is the best tempered and most harmonious system that the nature of sounds is capable of? (*x*).

14. It may not be amiss to observe that in Fig. 44, 45, 46, *Ec — E_g*, the difference of the Arithmetical and Harmonical mean temperaments of the 111^d , computed for one octave is $\frac{1}{214}$, for two is $\frac{1}{98}$, for three is $\frac{1}{69}$ of a comma. Hence in 3 octaves the arithmetical and harmonical mean temperaments of the v^{th} s are as 76 to 77 very nearly, and if the bases of any v^{th} in each system be unisons, their beats made in equal times are also as 76 to 77 (*y*): whence I judge that the harmony of the sounds in the two systems can scarce be sensibly different (*z*). Nevertheless it appears by the demonstration of the proposition, that an accurate solution of it required the help of Harmonical Means.

(*t*) Prop. III.

(*u*) Sect. IV. Art. 7.

(*x*) See Scholium. Prop. III.

(*y*) Prop. XI. cor. 4.

(*z*) Prop. XI. schol. I. art. 4.

TAB. II. PART I.

N ^o	Ratios of the temperaments for equal harmony of the		
I	V and	VI as	5 : 2
2	V	3 ^d	5 : 1
3	4 th	VI	5 : 4
I	4 th	3 ^d	5 : 2
4	V	III	10 : 3
5	V	6 th	20 : 3
I	4 th	III	5 : 3
4	4 th	6 th	10 : 3
6	VI	III	4 : 3
7	VI	6 th	8 : 3
I	3 ^d	III	2 : 3
6	3 ^d	6 th	4 : 3
	In one	Octave	I dozen
2	V and	VI + VIII	5 : 1
8	V	3 ^d + VIII	10 : 1
I	4 th	VI + VIII	5 : 2
2	4 th	3 ^d + VIII	5 : 1
I	V	III + VIII	5 : 3
9	V	6 th + VIII	40 : 3
10	4 th	III + VIII	5 : 6
5	4 th	6 th + VIII	20 : 3
I	VI	III + VIII	2 : 3
II	VI	6 th + VIII	16 : 3
12	3 ^d	III + VIII	1 : 3
7	3 ^d	6 th + VIII	8 : 3
			2 dozen

TABLE I. facing p. 143.

Contains the characters and terms of the perfect ratios of all the concords.

1 st Parcel.	2 ^d Parcel.	3 ^d Parcel.
V. $\frac{2}{3}$	VI. $\frac{3}{5}$	III. $\frac{4}{5}$
4 th . $\frac{3}{4}$	3 ^d . $\frac{5}{6}$	6 th . $\frac{5}{8}$
V + VIII. $\frac{1}{3}$	VI + VIII. $\frac{3}{10}$	III + VIII. $\frac{2}{5}$
4 th + VIII. $\frac{3}{8}$	3 ^d + VIII. $\frac{5}{12}$	6 th + VIII. $\frac{5}{16}$
V + 2VIII. $\frac{1}{6}$	VI + 2VIII. $\frac{3}{20}$	III + 2VIII. $\frac{1}{5}$
4 th + 2VIII. $\frac{3}{16}$	3 ^d + 2VIII. $\frac{5}{24}$	6 th + 2VIII. $\frac{5}{32}$
V + 3VIII. $\frac{1}{12}$	VI + 3VIII. $\frac{3}{40}$	III + 3VIII. $\frac{1}{10}$
4 th + 3VIII. $\frac{3}{32}$	3 ^d + 3VIII. $\frac{5}{48}$	6 th + 3VIII. $\frac{5}{64}$
&c.	&c.	&c.
&c.	&c.	&c.

TAB. II. PART II.

N ^o	Reciprocals of the temperaments of the v, 4 th & Comp. vi, 3 ^d & Comp. iii, 6 th & Comp.		
1	3.40000	8.50000	5.66667
2	3.20000	16.00000	4.00000
3	3.80000	4.75000	19.00000
1	3.40000	8.50000	5.66667
4	3.70000	5.28571	12.33333
5	3.85000	4.52941	25.66667
1	3.40000	8.50000	5.66667
4	3.70000	5.28571	12.33333
6	3.57143	6.25000	8.33333
7	3.72727	5.12500	13.66667
1	3.40000	8.50000	5.66667
6	3.57143	6.25000	8.33333
Sums	42.72013	87.47583	126.33334
2	3.20000	16.00000	4.00000
8	3.10000	31.00000	3.44444
1	3.40000	8.50000	5.66667
2	3.20000	16.00000	4.00000
1	3.40000	8.50000	5.66667
9	3.92500	4.24324	52.33333
10	5.20000	2.36364	5.66667
5	3.85000	4.52941	25.66667
1	3.40000	8.50000	5.66667
11	3.84211	4.56250	24.33333
12	3.25000	13.00000	4.33333
7	3.72727	5.12500	13.66667
Sums	43.49438	122.32379	144.44445

TAB. II. PART I.

N°	Ratios of the temperaments for equal harmony of the		
2	V + VIII	VI	5 : 1
8	V + VIII	3 ^d	10 : 1
13	4 th + VIII	VI	5 : 8
3	4 th + VIII	3 ^d	5 : 4
5	V + VIII	III	20 : 3
9	V + VIII	6 th	40 : 3
10	4 th + VIII	III	5 : 6
1	4 th + VIII	6 th	5 : 3
1	VI + VIII	III	2 : 3
6	VI + VIII	6 th	4 : 3
12	3 ^d + VIII	III	1 : 3
1	3 ^d + VIII	6 th	2 : 3
			3 dozen
8	V + VIII	VI + VIII	10 : 1
14	V + VIII	3 ^d + VIII	20 : 1
3	4 th + VIII	VI + VIII	5 : 4
1	4 th + VIII	3 ^d + VIII	4 : 2
4	V + VIII	III + VIII	10 : 3
15	V + VIII	6 th + VIII	80 : 3
16	4 th + VIII	III + VIII	5 : 12
4	4 th + VIII	6 th + VIII	10 : 3
12	VI + VIII	III + VIII	1 : 3
7	VI + VIII	6 th + VIII	8 : 3
17	3 ^d + VIII	III + VIII	1 : 6
6	3 ^d + VIII	6 th + VIII	4 : 3
	In two	Octaves	4 dozen

TAB. II. PART II.

N ^o	Reciprocals of the temperaments of the		
	v, 4 th & Comp.	vi, 3 ^d & Comp.	iii, 6 th & Comp.
2	3.20000	16.00000	4.00000
8	3.10000	31.00000	3.44444
13	4.60000	2.87500	8.33333
3	3.80000	4.75000	19.00000
5	3.85000	4.52941	25.66667
9	3.92500	4.24324	52.33333
10	5.20000	2.36364	5.66667
1	3.40000	8.50000	5.66667
1	3.40000	8.50000	5.66667
6	3.57143	6.25000	8.33333
12	3.25000	13.00000	4.33333
1	3.40000	8.50000	5.66667
Sums	44.69643	110.51129	122.11111
8	3.10000	31.00000	3.44444
14	3.05000	61.00000	3.21053
3	3.80000	4.75000	19.00000
1	3.40000	8.50000	5.66667
4	3.70000	5.28571	12.33333
15	3.96250	4.11688	105.66667
16	6.40000	1.88235	3.33333
4	3.70000	5.28571	12.33333
12	3.25000	13.00000	4.33333
7	3.72727	5.12500	13.66667
17	3.14286	22.00000	3.66667
6	3.57143	6.25000	8.33333
Sums	44.80406	168.19565	188.98830

TAB. II. PART I.

N ^o	Ratios of the temperaments for equal harmony of the		
8	v	VI + 2VIII	10 : 1
14	v	3 ^d + 2VIII	20 : 1
2	4 th	VI + 2VIII	5 : 1
8	4 th	3 ^d + 2VIII	10 : 1
10	v	III + 2VIII	5 : 6
15	v	6 th + 2VIII	80 : 3
16	4 th	III + 2VIII	5 : 12
9	4 th	6 th + 2VIII	40 : 3
12	VI	III + 2VIII	1 : 3
18	VI	6 th + 2VIII	32 : 3
17	3 ^d	III + 2VIII	1 : 6
11	3 ^d	6 th + 2VIII	16 : 3
			5 dozen
14	v + VIII	VI + 2VIII	20 : 1
19	v + VIII	3 ^d + 2VIII	40 : 1
1	4 th + VIII	VI + 2VIII	5 : 2
2	4 th + VIII	3 ^d + 2VIII	5 : 1
1	v + VIII	III + 2VIII	5 : 3
20	v + VIII	6 th + 2VIII	160 : 3
21	4 th + VIII	III + 2VIII	5 : 24
5	4 th + VIII	6 th + 2VIII	20 : 3
17	VI + VIII	III + 2VIII	1 : 6
11	VI + VIII	6 th + 2VIII	16 : 3
22	3 ^d + VIII	III + 2VIII	1 : 12
7	3 ^d + VIII	6 th + 2VIII	8 : 3
			6 dozen

TAB. II. PART II.

N ^o	Reciprocals of the temperaments of the v, 4th & Comp. v1, 3d & Comp. 111, 6th & Comp.		
8	3. 10000	31. 00000	3. 44444
14	3. 05000	61. 00000	3. 21053
2	3. 20000	16. 00000	4. 00000
8	3. 10000	31. 00000	3. 44444
10	5. 20000	2. 36364	5. 66667
15	3. 96250	4. 11688	105. 66667
16	6. 40000	1. 88235	3. 33333
9	3. 92500	4. 24324	52. 33333
12	3. 25000	13. 00000	4. 33333
18	3. 91429	4. 28125	45. 66667
17	3. 14286	22. 00000	3. 66667
11	3. 84211	4. 56250	24. 33333
Sums	46. 08676	195. 44986	243. 09941
14	3. 05000	61. 00000	3. 21053
19	3. 02500	121. 00000	3. 10256
1	3. 40000	8. 50000	5. 66667
2	3. 20000	16. 00000	4. 00000
1	3. 40000	8. 50000	5. 66667
20	3. 98125	4. 05732	212. 33333
21	8. 80000	1. 51724	2. 16667
5	3. 85000	4. 52941	25. 66667
17	3. 14286	22. 00000	3. 66667
11	3. 84211	4. 56250	24. 33333
22	3. 07692	40. 00000	3. 33333
7	3. 72727	5. 12500	13. 66667
Sums	46. 49541	296. 79147	302. 81310

TAB. II. PART I.

N ^o	Ratios of the temperaments for equal harmony of the		
1	V + 2VIII	VI	5 : 2
2	V + 2VIII	3 ^d	5 : 1
23	4 th + 2VIII	VI	5 : 16
13	4 th + 2VIII	3 ^d	5 : 8
4	V + 2VIII	III	10 : 3
5	V + 2VIII	6 th	20 : 3
16	4 th + 2VIII	III	5 : 12
10	4 th + 2VIII	6 th	5 : 6
12	VI + 2VIII	III	1 : 3
1	VI + 2VIII	6 th	2 : 3
17	3 ^d + 2VIII	III	1 : 6
12	3 ^d + 2VIII	6 th	1 : 3
			7 dozen
2	V + 2VIII	VI + VIII	5 : 1
8	V + 2VIII	3 ^d + VIII	10 : 1
13	4 th + 2VIII	VI + VIII	5 : 8
3	4 th + 2VIII	3 ^d + VIII	5 : 4
1	V + 2VIII	III + VIII	5 : 3
9	V + 2VIII	6 th + VIII	40 : 3
21	4 th + 2VIII	III + VIII	5 : 24
1	4 th + 2VIII	6 th + VIII	5 : 3
17	VI + 2VIII	III + VIII	1 : 6
6	VI + 2VIII	6 th + VIII	4 : 3
22	3 ^d + 2VIII	III + VIII	1 : 12
1	3 ^d + 2VIII	6 th + VIII	2 : 3
			8 dozen

TAB. II. PART II.

N ^o	Reciprocals of the temperaments of the v, 4 th & Comp. vi, 3 ^d & Comp. iii, 6 th & Comp.		
1	3.40000	8.50000	5.66667
2	3.20000	16.00000	4.00000
23	6.20000	1.93750	3.18182
13	4.60000	2.87500	8.33333
4	3.70000	5.28571	12.33333
5	3.85000	4.52941	25.66667
16	6.40000	1.88235	3.33333
10	5.20000	2.36364	5.66667
12	3.25000	13.00000	4.33333
1	3.40000	8.50000	5.66667
17	3.14286	22.00000	3.66667
12	3.25000	13.00000	4.33333
Sums	49.59286	99.87361	48.18182
2	3.20000	16.00000	4.00000
8	3.10000	31.00000	3.44444
13	4.60000	2.87500	8.33333
3	3.80000	4.75000	19.00000
1	3.40000	8.50000	5.66667
9	3.92500	4.24324	52.33333
21	8.80000	1.51724	2.16667
1	3.40000	8.50000	5.66667
17	3.14286	22.00000	3.66667
6	3.57143	6.25000	8.33333
22	3.07692	40.00000	3.33333
1	3.40000	8.50000	5.66667
Sums	47.41621	154.13548	101.61111

TAB. II. PART I.

N ^o	Ratios of the temperaments for equal harmony of the		
8	v + 2VIII	VI + 2VIII	10 : 1
14	v + 2VIII	3 ^d + 2VIII	20 : 1
3	4 th + 2VIII	VI + 2VIII	5 : 4
1	4 th + 2VIII	3 ^d + 2VIII	5 : 2
10	v + 2VIII	III + 2VIII	5 : 6
15	v + 2VIII	6 th + 2VIII	80 : 3
24	4 th + 2VIII	III + 2VIII	5 : 4 ⁸
4	4 th + 2VIII	6 th + 2VIII	10 : 3
22	VI + 2VIII	III + 2VIII	1 : 12
7	VI + 2VIII	6 th + 2VIII	8 : 3
25	3 ^d + 2VIII	III + 2VIII	1 : 24
6	3 ^d + 2VIII	6 th + 2VIII	4 : 3
	In three	Octaves	9 dozen

TAB. III.

The numbers and sums of the -----		
Dozen	N8	v, 4 th & Comp.
In 1 VIII ^{ve} . 1 st	12	42.72013
2 ^d	12	43.4943 ⁸
3 ^d	12	44.69643
4 th	12	44.80406
In 2 VIII ^{ves} .	4 ⁸	175.71500
5 th	12	46.08676
6 th	12	46.49541
7 th	12	49.59286
8 th	12	47.41621
9 th	12	53.22812
In 3 VIII ^{ves} .	108	418.53436

TABLE IV. Contains all the different temperaments for making every two concords, in the different parcels in 3 Octaves, equally harmonious.

N ^o	V 4 th &c	VI 3 ^d &c	Iproccals of all the different temperaments 63 Octaves. 8 ^d & Comp.			III, 6 th & Comp.	
1	$\frac{5}{17}^c$	$\frac{2}{17}^c$	7	8.50000	3)	17	5.66667
			6	16.00000	4)	16	4.00000
2	$\frac{5}{16}$	$\frac{1}{16}$	9	4.75000	1)	19	19.00000
			7	5.28571	3)	37	12.33333
3	$\frac{5}{15}$	$\frac{4}{15}$	7	4.52941	3)	77	25.66667
			5	6.25000	3)	25	8.33333
4	$\frac{10}{14}$	$\frac{7}{14}$	1	5.12500	3)	41	13.66667
				Octave			
5	$\frac{20}{13}$	$\frac{17}{13}$	1	31.00000	9)	31	3.44444
			7	4.24324	3)	157	52.33333
6	$\frac{7}{12}$	$\frac{4}{12}$	6	2.36364	— 6)	26	5.66667
			3	4.56250	3)	73	24.33333
7	$\frac{11}{11}$	$\frac{8}{11}$	3	13.00000	3)	13	4.33333
			3	2.87500	— 3)	23	8.33333
8	$\frac{41}{10}$	$\frac{41}{10}$	2	61.00000	19)	61	3.21053
			2	4.11688	3)	317	105.66667
9	$\frac{40}{31}$	$\frac{31}{31}$	1	1.88235	— 12)	32	3.33333
			1	22.00000	6)	22	3.66667
10	$\frac{157}{26}$	$\frac{11}{26}$	7	Octaves			
			7	4.28125	3)	137	45.66667
11	$\frac{19}{73}$	$\frac{16}{73}$	7	121.00000	39)	121	3.10256
			7	4.05732	3)	637	212.33333
12	$\frac{4}{13}$	$\frac{1}{13}$	7	1.51724	— 24)	44	2.16667
			7	40.00000	12)	40	3.33333
13	$\frac{5}{23}$	$\frac{8}{23}$	7	1.93750	— 11)	31	3.18182
			7	1.28302	— 48)	68	2.58333
			7	76.00000	24)	76	3.16667
				Octaves			

TABLE IV. Contains all the different temperaments for making every two concords, in the different parcels in 3 Octaves, equally harmonious.											
N ^o	v 4 th &c			N ^o	v 4 th &c						
	v	vi	iii		v	vi	iii				
	4 th	3 ^d	6 th		4 th	3 ^d	6 th				
	&c	&c	&c		&c	&c	&c				
1	5 ^c 17	2 ^c 17	3 ^c 17	14	20	1	19				
2	5	1	4	15	80	77	3				
3	16			16	317						
4	5	4	1	17	5	17	—12				
5	19			18	32						
6	10	7	3	19	7	1	6				
7	37			20	22						
8	20	17	3	21	In two Octaves						
9	77			22	35	32	3				
10	7	4	3	23	137						
11	25			24	40	1	39				
12	11	8	3	25	121						
13	41			26	160	157	3				
	In one Octave			27	637						
14	10	1	9	28	5	29	—24				
15	31			29	44						
16	40	37	3	30	13	1	12				
17	157			31	40						
18	5	11	—6	32	5	16	—11				
19	26			33	31						
20	19	16	3	34	5	53	—48				
21	73			35	68						
22	4	1	3	36	25	1	24				
23	13			37	76						
24	5	8	—3		In three Octaves						
25	23										

TABLE V. Contains the reciprocals of all the different temperaments in 3 Octaves.											
N ^o	v, 4 th & Comp.			N ^o	vi, 3 ^d & Comp.			N ^o	iii, 6 th & Comp.		
	v	4 th	Comp.		vi	3 ^d	Comp.		iii	6 th	Comp.
1	5)	17	3.40000	1	2)	17	8.50000	3)	17	5.66667	
2	5)	16	3.20000	2	1)	16	16.00000	4)	16	4.00000	
3	5)	19	3.80000	3	4)	19	4.75000	1)	19	19.00000	
4	10)	37	3.70000	4	7)	37	5.28571	3)	37	12.33333	
5	20)	77	3.85000	5	17)	77	4.52941	3)	77	25.66667	
6	7)	25	3.57143	6	4)	25	6.25000	3)	25	8.33333	
7	11)	41	3.72727	7	8)	41	5.12500	3)	41	13.66667	
	In	one	Octave								
8	10)	31	3.10000	8	1)	31	31.00000	9)	31	3.44444	
9	40)	157	3.92500	9	37)	157	4.24324	3)	157	52.33333	
10	5)	26	5.20000	10	11)	26	2.36364	—	6)	26	5.66667
11	19)	73	3.84211	11	16)	73	4.56250	3)	73	24.33333	
12	4)	13	3.25000	12	1)	13	13.00000	3)	13	4.33333	
13	5)	23	4.60000	13	8)	23	2.87500	—	3)	23	8.33333
14	20)	61	3.05000	14	1)	61	61.00000	19)	61	3.21053	
15	80)	317	3.96250	15	77)	317	4.11688	3)	317	105.66667	
16	5)	32	6.40000	16	17)	32	1.88235	—	12)	32	3.33333
17	7)	22	3.14286	17	1)	22	22.00000	6)	22	3.66667	
	In	two	Octaves								
18	35)	137	3.91429	18	32)	137	4.28125	3)	137	45.66667	
19	40)	121	3.02500	19	1)	121	121.00000	39)	121	3.10256	
20	160)	637	3.98125	20	157)	637	4.05732	3)	637	212.33333	
21	5)	44	8.80000	21	29)	44	1.51724	—	24)	44	2.16667
22	13)	40	3.07692	22	1)	40	40.00000	12)	40	3.33333	
23	5)	31	6.20000	23	16)	31	1.93750	—	11)	31	3.18182
24	5)	68	13.60000	24	53)	68	1.28302	—	48)	68	2.58333
25	25)	76	3.04000	25	1)	76	76.00000	24)	76	3.16667	
	In	three	Octaves								

TAB. II. PART II.

N ^o	Reciprocals of the temperaments of the		
	v, 4 th & Comp.	vi, 3 ^d & Comp.	iii, 6 th & Comp.
8	3. 10000	31. 00000	3. 44444
14	3. 05000	61. 00000	3. 21053
3	3. 80000	4. 75000	19. 00000
1	3. 40000	8. 50000	5. 66667
10	5. 20000	2. 36364	5. 66667
15	3. 96250	4. 11688	105. 66667
24	13. 60000	1. 28302	2. 58333
4	3. 70000	5. 28571	12. 33333
22	3. 07692	40. 00000	3. 33333
7	3. 72727	5. 12500	13. 66667
25	3. 04000	76. 00000	3. 16667
6	3. 57143	6. 25000	8. 33333
Sums	53. 22812	245. 67425	172. 07164

TAB. III.

----- reciprocals of the temperaments	
vi, 3 ^d & Comp.	iii, 6 th & Comp.
87. 47583	126. 33334
122. 32379	144. 44445
110. 51129	122. 11111
168. 19565	188. 98830
488. 50656	581. 87720
195. 44986	243. 09941
296. 79147	302. 81310
99. 87361	48. 18182
154. 13548	101. 61111
245. 67425	172. 07164
1480. 43123	1449. 65428

TAB. VI.

*The values of Eq and Eq' in Fig. 45 and 46, ---
wards the temperament of the 111^d, for ---
equally and the most harmonious.*

In 1 Octave.

$$(a) \quad \frac{12}{42.72013} = 0.2808980 = GD = d$$

$$\frac{12}{87.47583} = 0.1371808 = AH = b$$

$$\frac{12}{126.33333} = 0.0949868 = EM = m$$

In 2 Octaves.

$$\frac{48}{175.71500} = 0.2731696 = GD = d$$

$$\frac{48}{488.50656} = 0.0982587 = AH = b$$

$$\frac{48}{581.87720} = 0.0824916 = EM = m$$

In 3 Octaves.

$$\frac{108}{418.53436} = 0.2580433 = GD = d$$

$$\frac{108}{1480.43123} = 0.0729517 = AH = b$$

$$\frac{108}{1449.65428} = 0.0745005 = EM = m$$

(a) See the last Table.

TAB. VI.

--- being the first and second approximations to-
 --- making all the concords in 1, 2 or 3 octaves

Hence

$$Ef = 4d - 1 = 0.1235920$$

$$Ei = \frac{1-4b}{3} = 0.1504256$$

$$EM = m = 0.0949868$$

$$3 \mid 0.3690044$$

$$Eq = 0.1230015$$

$$Eq' = 0.122233. \text{ in 1 Octave,}$$

See Tab. VII.

$$Ef = 4d - 1 = 0.0926784$$

$$Ei = \frac{1-4b}{3} = 0.2023217$$

$$EM = m = 0.0824916$$

$$3 \mid 0.3774917$$

$$Eq = 0.1258306$$

$$Eq' = 0.124719. \text{ in 2 Octaves.}$$

$$Ef = 4d - 1 = 0.0321732$$

$$Ei = \frac{1-4b}{3} = 0.2360634$$

$$Em = m = 0.0745005$$

$$3 \mid 0.3427371$$

$$Eq = 0.1142457$$

$$Eq' = 0.11024.. \text{ in 3 Octaves.}$$

TAB. VII.

*The computation of Eq' in Fig. 46, being the ---
ment of the III^d, for making all the concords ---*

$$\frac{1}{GD} = \frac{1}{d} = \frac{42.72013}{12} = 3.560011$$

$$\frac{1}{Gg} = \frac{3}{1-b} = \frac{3}{0.8628192} = 3.476974$$

$$\frac{1}{Gk} = \frac{4}{1+m} = \frac{4}{1.0949868} = 3.652913$$

$$3) 10.689898$$

$$\text{Arith. mean } 3.563299$$

$$GD' = d' = \frac{1}{3.563299} = 0.280631$$

$$\frac{1}{AH} = \frac{1}{b} = \frac{87.47583}{12} = 7.289653$$

$$\frac{1}{Ae} = \frac{1}{1-3d} = \frac{1}{0.1573060} = 6.357031$$

$$\frac{1}{Al} = \frac{4}{1-3m} = \frac{4}{0.7150396} = 5.494096$$

$$3) 19.240780$$

$$\text{Arith. mean } 6.413593$$

$$AH' = b' = \frac{1}{6.413593} = 0.155919$$

TAB. VII.

--- second approximation towards the tempera-
 --- in 1 octave equally and the most harmonious.

$$\frac{I}{EM} = \frac{I}{m} = \frac{126.33333}{12} = 10.527777$$

$$\frac{I}{Ef} = \frac{I}{4d-1} = \frac{I}{0.1235920} = 8.091138$$

$$\frac{I}{Ei} = \frac{3}{1-4b} = \frac{3}{0.4512768} = \underline{6.647805}$$

$$3) 25.266720$$

$$\text{Arith. mean } 8.422240$$

$$EM' = m' = \frac{I}{8.422240} = 0.118733$$

$$\text{Hence } Ef' = 4d' - 1 = 0.122524$$

$$Ei' = \frac{1-4b'}{3} = 0.125441$$

$$Em' = m' = \underline{0.118733}$$

$$3) 0.366698$$

$$Eq' = 0.122233$$

See the values of Eq' in 2 and 3 octaves in Tab. VI,
 part 2^d.

PRO-

PROPOSITION XVII.

A system of commensurable intervals deduced from dividing the octave into 50 equal parts, and taking the limma $L = 5$ of them, the tone $T = 8$ and consequently the lesser $3^d L + T = 13$, the greater $111^d 2 T = 16$, the $4^{th} L + 2 T = 21$, the $5^{th} L + 3 T = 29$, &c, according to the table of elements (a), will differ insensibly from the system of equal harmony: I mean with regard to the harmony of the respective consonances in both.

For since $L = 5$, $T = 8$ and the $111^d 2 T = 16$ and the $VIII = 5 T + 2 L = 50$, we have the $111^d 2 T : VIII :: 8 : 25$; whence the $111^d 2 T = \frac{8}{25} VIII = \frac{8}{25} \log. 2 = \frac{8}{25} \times 0.30102.99957 = 0.09632.95962$, which subtracted from the perfect $111^d = \log. \frac{4}{5} = 0.09691.00130$, leaves the temperament $0.00058.04168$, which is to the comma $c = \log. \frac{81}{80} = 0.00539.50319$ as 4 to 37 very nearly (b). Hence the
tem-

(a) Prop. III.

(b) See an example of the like reduction in the next Scholium.

temperament of the III^d is $-\frac{4}{37}c$, and those of the vth and vith as in Tab. I, by prop. III. cor. 1. 2. 3.

TABLE I.

T : L :: 8 : 5 VIII = 50	The system of equal har- mony.	Ratios of the temperaments and of the beats made in any given time (c).
v $-\frac{1}{4}c - \frac{1}{37}c$	v $-\frac{1}{4}c - \frac{1}{36}c$	$\frac{1}{4} + \frac{1}{37} : \frac{1}{4} + \frac{1}{36} :: 369 : 370$
vi $+\frac{1}{4}c - \frac{3}{37}c$	vi $+\frac{1}{4}c - \frac{3}{36}c$	$\frac{1}{4} - \frac{3}{37} : \frac{1}{4} - \frac{3}{36} :: 75 : 74$
III $-\frac{4}{37}c$	III $-\frac{4}{36}c$	$\frac{4}{37} : \frac{4}{36} :: 36 : 37$

Now though the concords of the same name in this system and that of equal harmony are not exactly equally harmonious (*d*), and though the difference of an unit in the largest number of beats made in a given time may be distinguished by counting them; yet if the numbers be not smaller than those in the table, the difference in the harmony of the concords will be deemed insensible by proper judges; which are those only that have carefully attended to the beats of concords in tuning instruments. But any one else may be satisfied experimentally, by
causing

(c) Prop. XI. coroll. 4.

(d) Prop. XIII. coroll. 5.

causing two concords to the same base to beat as in the table. Q. E. D.

Scholium.

In like manner if $T = 5$ and $L = 3$, then the octave $5T + 2L$ is $= 31$ and the temperament of this system, which *Hugenius* has adopted (*e*), will be found as in the third column of the next table.

TABLE II.

$T:L::2:1$ VIII = 12	$T:L::3:2$ VIII = 19	$T:L::5:3$ VIII = 31
$v - \frac{1}{4}c + \frac{3}{19}c$	$v - \frac{1}{4}c - \frac{3}{35}c$	$v - \frac{1}{4}c + \frac{1}{110}c$
$vi + \frac{1}{4}c + \frac{9}{19}c$	$vi + \frac{1}{4}c - \frac{9}{35}c$	$vi + \frac{1}{4}c + \frac{3}{110}c$
III $+ \frac{12}{19}c$	III $- \frac{12}{35}c$	III $+ \frac{4}{110}c$

On the contrary, if from the given temperament of a system it be required to find the ratio of T to L , we may proceed as follows. Let it be proposed to approximate to the system of equal harmony, where $2T = III - \frac{1}{9}c$ (*f*);
then

(*e*) *Cyclos harmonicus*, at the end of his Works, or *Histoire des Ouvrages des Sçavans*, Octob. 1691, pag. 78.

(*f*) Prop. xvi. Scholium 2, Art. 9.

then since $5T + 2L = \text{VIII}$, we have $2L =$
 $(\text{VIII} - 5T) = \text{VIII} - \frac{5}{2} \times \text{III} - \frac{1}{9}c$, whence
 $T : L :: \text{III} - \frac{1}{9}c : \text{VIII} - \frac{5}{2} \times \text{III} - \frac{1}{9}c$.

To find this ratio, we have the $\text{III} = \log. \frac{5}{4}$
 $= 0.09691.00130$ and the comma $c = \log. \frac{81}{80}$
 $= 0.00539.50319$ and $\frac{1}{9}c = 0.00059.94480$.
 Whence $2T = \text{III} - \frac{1}{9}c = 0.09631.05650$
 and $\frac{5}{2} \times \text{III} - \frac{1}{9}c = 0.24077.64125$ and the
 $\text{VIII} = \log. 2 = 0.30102.99957$ and $2L =$
 $\text{VIII} - \frac{5}{2} \times \text{III} - \frac{1}{9}c = 0.06025.35832$, and
 lastly $T : L :: 963105650 : 602535832$.

Now the quotients of the greater term of this
 ratio divided by the lesser and of the lesser di-
 vided by the remainder and of the former re-
 mainder by the latter &c, are 1, 1, 1, 2, 24, &c.
 Whence the ratios greater than the true one are
 2 to 1, 5 to 3, 8 to 5, &c, and the lesser are
 3 to 2, 11 to 7, &c (g).

Hence taking T to L successively in those ra-
 tios, by the method used in the demonstration
 of the proposition, the temperaments of the ap-
 proximating rational systems will be found as in
 the tables. By which we see how much and
 which

(g) See Mr. Cotes's *Harmonia Mensurarum*, Schol. 3.
 prop. 1.

which way they differ from that of mean tones, as well as from that of equal harmony in Table I.

SECTION VIII.

The scale of musical sounds is fully explained and made changeable upon the harpsichord, in order to play all the flat and sharp sounds, that are used in any piece of music, upon no other keys than those in common use.

DEFINITIONS.

I. The interval of a perfect octave being divided, in any tempered system, into 5 equal tones and 2 equal limmas (*b*), the excess of the tone above the limma is called a Minor limma.

II. The difference of the major and minor limma is called a Diesis.

III. If the difference of the intervals of two consonances to the same base be a diesis, I shall call either of them a False consonance when ever, in playing on the organ or harpsichord, it is substituted for the other which ought to be used; as it often is for want of a complete scale of sounds in those instruments.

IV. The notes A^* , B^* , &c, signify sounds which are sharper, and A^b , B^b , &c, sounds which

(*b*) See sect. IV art. 3, or the dem. of prop. 2, or prop. 3.

which are flatter by a minor limma than the respective primary sounds $A, B, \&c$: And A^{**} , A^{bb} , $\&c$, signify sounds whose distance from A is double the distance of A^* or A^b from A and alike situated.

1. Pl. XVIII. Fig. 48 or 49. The interval of a perfect octave being represented by the circumference of any circle (i) and supposed to be divided by the sounds A, B, C, D, E, F, G into 5 tones and 2 limmas, towards the acuter sounds take the interval AA^* equal to the minor limma $AB-BC$, and towards the graver take AA^b equal to AA^* , and when the like flat and sharp sounds are placed at that distance on each side of the other primary sounds B, C, D, E, F, G , every tone will be divided by a flat or a sharp sound into a major and a minor limma, and by both into two minor limmas with a diesis between them; and each primary limma, BC, EF , will be divided by a flat or a sharp sound into a minor limma and a diesis, and by both into two dieses with an interval between them.

2. Fig. 48. In the *Hugenian* system the octave is divided into 31 equal parts, of which the tone is 5, the major limma 3, the minor 2 and the diesis 1 (k).

Fig. 49. In the system of Equal Harmony the octave is divided into 50 equal parts, of which the tone is 8, the major limma 5, the minor 3 and the diesis 2 (l).

Therefore the former tone is to the latter as

$$\begin{array}{r} L \\ 5 \\ \hline 31 \end{array}$$

(i) Sect. IV. art. 7.

(k) Prop. XVII, schol.

(l) Prop. XVII.

$\frac{5}{31}$ VIII to $\frac{8}{50}$ VIII, or as 125 to 124, and the former diesis is to the latter as $\frac{1}{31}$ to $\frac{2}{50}$, or 25 to 31; and since a quarter of a comma is about $\frac{1}{223}$ VIII (*m*) the former diesis $\frac{1}{31}$ VIII contains above $\frac{7}{4}$, and the latter almost $\frac{9}{4}$ of a comma.

3. Fig. 48 or 49. In either of those systems or any other of that kind, by going many times round the circle it will appear, that in ascending from *F* continually by v^{ths} the 7 primary notes will first occur in this order *FCGDAEB*, and then recur once sharpened in the same order, and again twice sharpened &c: Likewise in descending from *F* by v^{ths} , they will recur once flattened in that order thus inverted, *B^b E^b A^b D^b G^b C^b F^b*, and again twice flattened &c: And these several cycles joined together make the following progression ascending by v^{ths} ; *E^{bb} B^{bb}, F^b C^b G^b D^b A^b E^b B^b, FCGDAEB, F* C* G* D* A* E* B*, F** C** G** D** A** E** B**, F*** C*** G*** D*** E*** B**** &c.

4. Hence a Table of the minor and major consonances to any number of Keys or base notes in that progression placed in the first column (*n*), is thus deduced. Opposite to any Key as *D* write the 12 trebles *E^b, E, F, F*, &c* of the minor and major consonances within the VIII in the order of their marks, 2^d, 11^d, 3^d, 111^d, &c at the top of the table, which trebles are found by going round the circle; then place the same progression of v^{ths} above

(*m*) Found by dividing the log. of 2 by $\frac{1}{4} \log. \frac{81}{80}$.

(*n*) Plate XIX.

above and below the treble E^b in col. 2, as stands above and below E^b in col. 1; and having done the like to the other trebles E , F , &c, the table is finished.

For since the interval DA in col. 1 is equal to $E^b B^b$ in col. 2, it follows that in col. 1 and 2 the interval AB^b equals DE^b ; and the same may be said of the rest of the table. At the bottom of it the letters L , l , D signify the major and minor limma and the diesis, as being the differences of the intervals marked at the top.

5. As the organ or harpsicord has but 12 sounds in the octave, whose notes are F, C, G, D, A, E, B , with F^* , C^* , G^* , above, and E^b , B^b , below them in col. 1; all the notes below E^b in col. 1 and in those of the minor consonances, and all above G^* in col. 1 and in those of the major consonances have no sounds answering to them in those instruments; and are therefore excluded, or distinguished from the notes that have sounds, by circles round them, both in the table and in Fig. 48 and 49.

Consequently when any of the excluded notes D^* , A^* , E^* , B^* , F^{**} , C^{**} , that are above G^* , occur in a piece of music, as most of them often do, the musician is obliged to substitute for them the sounds of E^b , B^b , F , C , G , D , respectively, which being higher by a diesis (\circ) make false consonances (p).

L 2

Likewise

(\circ) As appears by Fig. 48, or by the collateral notes in the columns of 14ths and 5ths in the table of consonances.

(p) Def. III.

Likewise when any of the excluded notes A^b , D^b , G^b , C^b , F^b , B^{bb} , that are below E^b , occur, as some of them often do, the musician must substitute for them G^* , C^* , F^* , B , E , A , respectively, which being lower by a diesis make false consonances.

Hence the two middlemost Keys D , A have one false consonance in each, and the numbers of them in the successive higher or lower Keys, increase in the arithmetical progression 2, 3, 4, 5, 6. Whence it is easy to collect that seven twenty-fourths of the whole number of major and minor consonances in the scale of the organ or harpsichord, are false; besides a larger proportion of false ones among the Superfluous and Diminished consonances hereafter mentioned.

6. The consonances to all the Keys above E have no flat notes; because B^b is the highest flat note in every column of minor consonances, and is the highest of all where it is the minor 5th to the Key E . Again, the consonances to all the keys below C have no sharp notes; because F^* is the lowest sharp note in every column of major consonances, and is the lowest of all where it is the major 1vth to the Key C . Therefore the consonances to those two and the intermediate Keys, $CGDAE$, have both flat and sharp notes among them.

Hence it comes to pass that the concords (q) to the 4 middlemost keys G , D , A , E , which are the

(q) Sect. III. art. 11th.

the open strings of the violin, are all true, but not all the discords.

7. By adding a sound for A^b , every one of the 6 lower keys E^b, B^b, F, C, G, D , will have one false consonance changed into a true one, as appears by inspection of the oblique diagonal rows of A^b in the table. Likewise by adding another sound for D^* every one of the 6 higher keys A, E, B, F^*, C^*, G^* , will have one false consonance changed into a true one. Now in this enlarged scale of 14 keys all the consonances to D and A , the two middlemost, are true. And a like advantage will follow from giving sounds to D^b and A^* , the two next exterior keys, and so forth.

Therefore universally, the number of the middlemost keys to which all the minor and major consonances are true, is equal to the whole number of keys or sounds in the octave diminished by 12; so that the 24 sounds in col. 1. would be necessary to make all these consonances true in the 12 middlemost keys.

8. But besides the major and minor consonances in the Table there are others in the scale of Fig. 48 or 49, which I think are called Superfluous and Diminished consonances.

The interval of a major consonance augmented by a minor limma makes the interval of a superfluous consonance; and the interval of a minor consonance diminished by a minor limma makes the interval of a diminished consonance.

Thus the treble of a superfluous II^d , III^d , IV^{th} , V^{th} , VI^{th} , VII^{th} , &c to the key F , is G^* , A^* , B^* , C^* , D^* , E^* , respectively; and the treble of the diminished 2^d , 3^d , 4^{th} , 5^{th} , 6^{th} , 7^{th} , &c to the key B , is C^b , D^b , E^b , F^b , G^b , A^b . And the like is to be understood in the other keys, where the trebles are often double sharp and double flat sounds; but are all omitted in the Table to avoid confusion by adding so many notes to it.

9. I have heard of but one method of supplying the organ or harpsichord with more sounds in each octave; which is by adding pipes or strings for A^b , D^b , &c, and dividing the keys of their substitutes G^* , C^* , &c, each into two keys, the longer of them for sounding G^* , C^* , &c as usual, and the shorter for sounding A^b , D^b , &c: and by doing the like for D^* , A^* , &c. But this method of supplying the defects of the scale is quite laid aside, on account of the great difficulty in playing upon so many keys without extraordinary practice, and the following palliative remedy is universally received.

Pl. XVIII. Fig. 48 or 49. The octave being always divided into 5 tones and two limmas; by increasing the tones equally till each becomes double the diminishing limma BC or EF , the diesis, or difference between the major and minor limma, will be contracted to nothing, which by Defin. III annihilates all the false consonances. But the harmony in this system of 12 Hemitones is extremely coarse and disagreeable.

For

For the temperaments of the v^{th} and 4^{th} , vi^{th} and 3^{d} , iii^{d} and 6^{th} and their compounds with $viii^{\text{ths}}$, are nearly $\frac{1}{10}$, $\frac{7}{10}$ and $\frac{6}{10}$ of a comma respectively (r) and in the system of equal harmony they are $\frac{5}{18}$, $\frac{1}{6}$ and $\frac{1}{9}$ (s); by which system, as being the most harmonious, all other systems ought to be examined, as by a standard. Now $\frac{1}{10}$ being much less than $\frac{5}{18}$, makes the concords in the first parcel (t) finer than they ought to be; and $\frac{7}{10}$ and $\frac{6}{10}$ being much greater than $\frac{1}{6}$ and $\frac{1}{9}$, make the concords in the other two parcels much coarser than they ought to be, the two least of those temperaments being as great as those concords can properly bear.

Now for want of another sound to terminate each diesis in the scale, it is necessary in the tuning to diminish the diesis till one sound may serve tolerably for the other, and thus to approach towards that inharmonious system of 12 hemitones, till the harmony of the scale becomes very coarse before the false consonances are barely tolerable (u).

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9. That

(r) Prop. xvii. Tab. ii^d. col. 1.

(s) Prop. xvi. schol. 2. art. 10 and 13.

(t) Prop. iii. schol.

(u) This is done by sharpening the major iii^{ds} more than the ear can well bear, which enlarges the tones and lessens the major limmas and diesis: or, because any 6 tones or 3 major iii^{ds} and a diesis (as $A^b C + CE + EG * + G * A^b$) make up the octave or circumference in Fig. 48.

9. That this is a bad expedient for supplying the want of more sounds, is farther evident from the *Hugenian* system, where the temperament common to the vith and 3^d, &c being $\frac{1}{4} + \frac{3}{110}$ of a comma (*x*) is considerably greater than it ought to be, that is, than $\frac{1}{6}$ of a comma, as in the system of equal harmony; and yet the *Hugenian* diesis is $\frac{7}{4}$ of a comma (*y*), which being considered as a temperament of the false consonances and being so much greater than $\frac{1}{4} + \frac{3}{110}$ of a comma must needs make horrible dissonance.

10. Having therefore been long dissatisfied with the coarseness of the harmony even of the true consonances in the scale of our present instruments, which is so defective too that not above a seventh or eighth part of the best compositions made since *Corelli's* time, nor above a third or fourth of his can be played upon it without using many false consonances; and being still more disgusted when these come into play, as they often do in the remaining two thirds or three fourths of *Corelli's* works, and six sevenths or seven eighths of all the rest; I was glad to find out a better remedy for both those defects; at least in a scale of single sounds.

11. The strings of the fore unison of the harpsichord being tuned as usual to the notes of the common scale in the following lower line, let the
sounds

(*x*) Prop. xvii. Tab. 2. col. 3.

(*y*) Sect. viii. art. 2.

sounds of the back unison be altered to the notes in the upper line, each of which differs from the note under it by a diesis (z).

$A^b B^{bb} A^* C^b B^* D^b C^{**} D^* F^b E^* G^b F^{**}$
 $G^* A B^b B C C^* D E^b E F F^* G$

Now since the jacks which strike the strings of any of these couples of notes, as G^* and A^b , stand both upon one key, by moving a stop hereafter described, that key can strike either string alone without sounding the other: And since both the sounds in any couple are seldom or never used in any single piece of music, the musician before he begins to play it, can put in, by the stop, that sound which he sees most occasion for; and either of them being struck by the same key, the execution is always the same as usual.

For example, if besides the sounds F^* , C^* , G^* in the common scale, D^* , A^* , E^* , B^* , F^{**} should also occur in a piece of music (a) move their stops, and their strings will be struck by the keys of $E^b B^b F C G$ respectively, whose sounds are usually substituted for the sounds required.

12. A musician by casting his eye over any piece of music, can soon see what flat or sharp sounds are used in it which are not in the common scale; and to save that trouble for the future, may write them down at the beginning of the piece. Now and then it may be proper to observe whether

(z) As appears by Fig. 49.

(a) As in *Corelli's* xith solo and *Carbone's* III^d &c.

ther the outermost of them in their progression by v^{th} s, should be put in or not, lest its substitute should occur oftener than the principal sound itself. If both occur, that which recurs oftener must be in the scale. But as both occur very seldom the matter is scarce worth notice.

13. To shew by inspection which are the false consonances in the Harpsichord after any flat or sharp sounds are put into it by the stops; imagine the two middlemost transverse parallelograms in the Table (*b*) and also the circles surrounding the notes which are not in the common harpsichord, to be drawn with the point of a diamond upon a pane of glass laid over them. Then if the sounds of D^* and A^* for instance be put into the harpsichord, move the pane two lines higher till the uppermost line of the two parallelograms just takes in those two notes in col. 1, and in this position the circles upon the pane will cover all those notes in the table which are not in the present scale of the harpsichord, and point out the false consonances to every key.

14. Thus you see how to make any given key as E or B , as free from false consonances as D or A is in the common fixed scale; namely by putting in by the stops as many sharp notes above G^* in the column of keys as shall bring E and B into the middle of the 12 keys then in instrument. And the like may be done for any given key below D by putting in flat notes below E^b .

And

(*b*) Plate XIX.

And thus a musician that transposes music at sight can accompany a voice with the purest and finest harmony in the properest key for the pitch of the voice. I say the finest harmony; because this changeable scale may easily be tuned to the most harmonious system (*c*) which is impracticable upon the common fixed scale, because the diesis would be so large as to render the false consonances insufferably bad (*d*).

15. The famous *Ruckers* and other musicians of a delicate ear, always valued the tone of a single string for its distinctness and clearness, spirit and duration, and preferred it to that of unisons and octaves. I must confess I have long been of that opinion, even before I thought of this changeable scale of single sounds, which however after some years experience upon my own harpsichord has fully confirmed me in it.

16. Unisons by themselves or with an octave are indeed an addition to the loudness of the tone, but not nearly in proportion to the number of strings. First because the oppression of the belly of the instrument by the force of so many strings, hinders the facility and duration of its tremors; and secondly because in tuning unisons or octaves, it is manifest that their tone is never clear, loud and flowing, like that of a single string, except when they are precisely perfect. But as this perfection continues but a very little time, especially after

(*c*) Prop. XVI. schol. 2. art. 10 and 13.

(*d*) Sect. VIII. art. 2.

after the room is warmed by company, that clear ringing tone is soon destroyed.

The compound tone of unisons by themselves, or with an octave, being of itself so indistinct, what beating and jarring must result from their complicated mixtures in playing three or four parts of music? Especially as the imperfections of the unisons and octaves in the course of playing are frequently added to the temperaments of the other consonances, which if they were perfect could not bear those imperfections so well as the unisons and octaves do when sounded by themselves (*e*). This confused noise, like that of a dulcimer, is but too plainly perceived when the ear is held over the strings of the harpsichord; and since it results from the multiplicity of strings, it appears that the best way to improve this instrument is to find out methods for increasing the strength and clearness of the tone of single strings.

17. To me, who seldom hear any other than the single strings of my own harpsichord, the tone is as loud as I desire, not only for lessons and cantatas but also concertos accompanied with instruments in a large room. This indeed is more than a person could expect who has seldom or never attended to the tone of single strings except in the short *pianos* after the long continued *fortes* upon the full harpsichord. The reason is that the very same objects affect our senses very differently in different circumstances, as is very evident in attending to any other sensation as well as that of sounds.

(*e*) Prop. XIII. coroll. 8.

sounds. “ For instance, in coming out of a strong
“ light into a room with the window-shutters al-
“ most closed, we immediately have a sensation
“ of darkness or a very little light, and this con-
“ tinues much longer than the pupil requires to
“ dilate and accommodate itself to that weak de-
“ gree of light, which is almost instantaneously
“ done. But after staying some time in the same
“ or a much darker place, the same room which
“ appeared dark before, will be sufficiently light.”
This observation is plainly applicable to sounds,
and more of them upon the other senses may be
seen in Dr. Jurin’s Essay on distinct and indistinct
Vision at the end of my Optics (f).

18. *An expedient for changing the sounds
of any harpsichord ready made, where-
by to experience the truth of the fore-
going observations.*

Pl. xxvi. The 66th figure represents the heads
t, u of two jacks standing as usual upon one
key, with their pens pointing opposite ways un-
der the strings on each side of them, as *G** and
A^b, the back unison being raised to *A^b*. And
abcd represents a small brass square of the size
in the figure, whose shorter leg *ab* is made very
thin and placed between the jacks with its flat
sides facing them; and the longer leg *bcde*, be-
ing placed directly over, and parallel to the next
couple of strings that are closest together, is filed
four

(f) Art. 267.

four square, and slides lengthways in two square notches at *c* and *d* made in the parallel sides *f c g*, *b d i* of a long brass plate turned up like the sides of a long shallow trough, which is supported a little above the strings by a row of small brass pillars placed between the larger intervals of the strings, as at *r*, *s*, &c, (but farther asunder) and skrewed fast into the pinboard of the harpsichord.

These pillars have long necks passing through the holes *r*, *s*, &c in the bottom of the trough, and the nuts *r*, *s*, &c are skrewed upon the necks down to the bottom, to hold it fast upon the shoulders of the pillars. And a brass lid *F G H I* with oblong holes *R S*, &c corresponding to *r*, *s*, &c, being laid upon the trough *f g b i*, the upper nuts *R*, *S*, &c must be skrewed upon the same necks, to keep the lid tightish upon the longer leg of the square *a b c d* and others of the same size. A slit *m n* is made in the lid for a short round pin *e* in the longer leg *c d* to come thro' it, and to move in it to and fro by a touch of the finger laid upon the pin. There must be as many such squares as keys or couples of jacks, and the trough and lid may be each of one piece or consist of two or three pieces joined together at the necks of the pillars or any where else.

While the jacks *t*, *u* are kept at their full height by holding down their key, with your finger laid upon the pin *e* push the leg *a b* against the far jack and mark the edge, or inner side of it with a line drawn close by the upper edge of the
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the leg ab ; and after the square is drawn back, make such another mark upon the edge of the near jack. Then from a small slender pin cut off a piece of a proper length measured from the point, and taking hold of its thicker end with a pair of pliers, press the point into the inner edge of the jack, a little above the mark and far enough to stick fast in it, and do the like to the opposite jack. Let each pin project from its jack about a quarter of the space between the two jacks, leaving about half of it void in the middle between the opposite ends of the pins, as represented in the figure.

Now when the two jacks are again raised by their key and kept at their full height, by drawing the square backwards with your finger laid upon the pin e in the longer leg, the shorter leg ab will come under the pin in the near jack, and keep it suspended with its pen above the string G^* , which therefore will be silent while the far jack plays alone upon the string A^b ; or, by pushing the square forward with your finger at e , the leg ab will go under the pin in the far jack, and suspend its pen above the string A^b , while the near jack plays alone upon the string G^* .

When all the strings of the back unison are tuned to the notes in the upper line in art 11th all their jacks must be suspended on the shorter legs of the squares; and then all the fore jacks will strike the sounds of the vulgar scale; and when other flat or sharp sounds are required in any piece of music, they must first be introduced by hold-
ing

ing down the keys of their usual substitutes, one by one, and by drawing back the corresponding squares with a finger laid upon their pins at *e*. So long as you choose to play upon this changeable scale, keep the knobs of the right-hand stops of a double harpsichord tyed together by a string.

When the strings are tuned unisons again, you may play upon them without removing this mechanism, provided you first draw every pin *e* towards the middle of the slit *mn*, in the lid *FI*, till it be opposite to the angular notch *o*, and then draw the lid lengthways by the button *p*, till the notch *o* embraces the pin *e* and keeps the shorter leg *ab* in the middle of the void space between the ends of the pins in the opposite jacks: otherwise these pins may sometimes strike against the shorter legs of the squares. If that middle space be too narrow, try whether it may not be widened a little by separating the sliders with some very thin wedges put between them: perhaps a little may be planed off from the back edges of the sliders without hurting them.

I have described this mechanism so fully, I think, that any man who works true in brass may easily apply it at a small expence to any harpsichord ready made, and take it quite away without the least damage to the instrument. I have used it some years in my own harpsichord with great pleasure and no other inconvenience than that of removing the music book in order to touch the pins in the brass squares behind it. But the following mechanism for the reception of which

a little preparation must be made in the fabric of a new harpsichord, is quite free from that inconvenience, and changes any sound together with all its octaves in an instant, without putting down their keys.

19. *To make a new harpsichord wherein the sounds being changeable at pleasure, the usual set of keys shall immediately strike the proper scale for any proposed piece of music.*

Pl. xxvi. Fig. 67. Conceiving the pins, strings and jacks which in every octave belong to the notes *A, B, D, E*, to be taken away from the fore unisons of a common harpsichord, the remaining pins, strings and jacks will be sufficient for the new harpsichord. Let the sounds of these strings be altered to the notes here placed by the sides of their pins, and let these notes be written on the pin board of the new harpsichord; and that the tones of the strings sounded by the jacks in each row, may be as like each other as possible, let the tongues of the new jacks be put as near as may be to their inner edges, and these opposite edges be placed in the new slider as near as may be to one another, as represented in the figure.

Each of the keys, *A, B, D, E*, that moves but one jack (which therefore must be made as heavy with lead as two of the other jacks) strikes always one and the same string. But each of the 8 remain-

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ing

ing keys, B^b , C , C^* , E^b , F , F^* , G , G^* , which moves a couple of jacks, is intended to strike either of their strings alone at pleasure; that is, A^* or B^b , B^* or C , C^* or D^b , D^* or E^b , E^* or F , F^* or G^b , F^{**} or G , G^* or A^b .

Pl. xxviii, Fig. 70. I think the best way to do this would be to have eight stops or brass knobs skrewed as usual on the shanks of eight draught irons made moveable in eight flits cut in the fore board of the new harpsichord: But to save a quarter of the labour and expence I propose to do it almost as well with only six, that is, three at each end of the fore board, as in the figure: where the notes of each couple of the changeable sounds are written on opposite sides of each knob, to the intent that the sound or string signified by this or that note *to which the knob is pushed*, may be struck alone by the key belonging to both the notes, while the other string is silent. And since eight sounds are intended to be changed by six knobs, each extreme knob is designed to change two sounds at one push of it towards either couple of notes at the end of the slit, according to the same rule as before.

Hence by pushing the two outermost knobs at the base end of the fore board, towards the right hand, and all the rest towards the left, the keys will strike the eight changeable sounds in the vulgar scale, namely F^* , C^* , G^* , E^b , B^b , F , C , G , to be occasionally changed by pushing the knobs the contrary way: the other four, A , B , D , E , are fixt sounds.

The

The notes of the changeable sounds are placed in such an order, that the sounds belonging to the notes on the same sides of the successive knobs, continually ascend or descend by v^{ths} , as in the Table of keys and consonances in Plate XIX. Because this order will be found much more convenient for altering and adapting the Scale to different pieces of music, than the alphabetical order of the same notes.

Now this design may be executed as follows.

Pl. XXVI. Fig. 67. When the pens for the back unisons are put under their strings, as denoted in the figure, and those for the fore unisons are drawn off from theirs by the stops of the common harpsichord, the jack holes in the two parallel rows have the same situation with respect to each other as they are intended to have in the new slider, except as already observed that the space between the two rows should be much narrower than in the common harpsichord.

By these directions if an accurate draught of all the jack holes be made upon a long brass plate, part of which draught is represented in the figure, it may serve as a general pattern for making the new sliders, or at least to give a clear conception of their dimensions, which a workman may execute in what manner he pleases. In order thereto let six brass plates well flatted by a mill be made equal to each other in all their dimensions. Let the length of each be equal to, or rather longer at first than that of a common slider, and the breadth of each be sufficient for

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leaving

leaving a pretty strong margin on the outsides of the jack holes, and then the thickness, after the work is finished and polished on both sides, need not exceed a twelfth of an inch.

Fig. 67. In the 1st plate or slider the opposite holes for the jacks A^* and B^b and all their octaves, being made equal to those in the general pattern, let all the rest be made wider on each side, than those in the pattern, by a twelfth of an inch, as represented at N^o 1, below fig. 67.

In the 2^d slider the holes for the jacks B^* and C and, to save another slider, for F^{**} and G being made equal to those in the pattern, let all the rest be made wider on each side, than those in the pattern, by $\frac{1}{12}$ of an inch, as at N^o 2.

In the 3^d slider the holes for the jacks C^* and D^b and, to save another slider, for F^* and G^b being made equal to those in the pattern, make all the rest wider on each side by $\frac{1}{12}$ of an inch.

In the 4th slider the holes for the jacks D^* and E^b , and in the 5th slider for E^* and F , and in the 6th slider for G^* and A^b being made equal to those in the pattern, let all the rest in each slider be made wider on each side, than those in the pattern, by $\frac{1}{12}$ of an inch.

Pl. xxvii. Fig. 68. An under socket pq being made of wood as usual, that is, with jack holes directly opposite to one another but nearer together as already observed, and being placed as near to the keys as may be, let an upper socket rs be made

made of brass, exactly equal to the pattern in fig. 67 but without the notches for the tongues to play in; in which socket let every jack hole except for *A*, *B*, *D*, *E*, be made wider on its left side only by $\frac{1}{12}$ of an inch.

Leave $\frac{6}{12}$ or half an inch in height above this upper socket *rs*, for the six sliders numbered 1, 2, 3, 4, 5, 6 as above, to lie upon it, and let all be supported as usual by cross pieces of boards fixed underneath.

All the jacks being put thro' their holes, this 68th figure represents a view of their edges and of the widening of the holes on each side, as they would appear to a distant eye placed at the fore end of the harpsichord, supposing the fore board and fore margin of the sockets and sliders were taken away.

When the two sockets *pq* and *rs* are so adjusted in their places that the jacks *A*, *B*, *D*, *E* stand upright and strike their strings, fix the sockets in that position at one end only, that the shrinking or swelling of the harpsichord may not bend or strain them. Then push all the sliders towards the right hand, till the pens on the right hand of the other jacks in the far row shall strike their strings too. see also fig. 67.

Then if any slider be drawn back again, which the widened holes will permit, it will draw back the jacks in its narrow holes only, without stirring the rest, and bring the right hand pen of the far

jack from under its string on the right hand and put the left hand pen of the near jack under its string on the left hand; and then this latter string will be sounded alone by the same key while the former is silent.

The holes in the sliders for the jacks *A, B, D, E*, which have no motion sideways, need be widened on their right sides only, as represented by the shades, to make room for the sliders to move towards the left hand; but if they be widened on both sides, according to the general direction above, no inconvenience will follow from it.

According to the usual breadth of harpsichords the compass of our scale may conveniently be from double *G* up to *e* in *alt*.

Pl. XXVIII. Fig. 69. When the six sliders are laid upon one another in any order, provided they coincide in length and breadth (and keep so by two pins put thro' two columns of holes at their ends) three round holes must be drilled through them all in the vacant places at *k* and *l*, opposite to the jacks *b* and *d* in *alt*, and at *m* a little above *e* in *alt*; and the hole at *k* in the 1st slider, at *l* in the 5th, as numbered above, and at *m* in the 2^d remaining round, all the rest must be lengthened by $\frac{1}{12}$ inch on the right hand and by as much on the left, to the end that a steel pin put thro' the round hole in any of those sliders (*g*) may draw it

(*g*) See those numbers in the lower line of Fig. 70.

it on either side $\frac{1}{12}$ of an inch, without moving any other slider.

Opposite to the centers of the round holes at k, l, m , and at the distance of about an inch and half from each center, are three other centers n, o, p upon the pin board, where two concentric circles are drawn about each, one with a radius about $\frac{3}{4}$, and the other about $\frac{1}{4}$ or $\frac{3}{10}$ of an inch. Each larger circle represents a brass plate having a cylindrical neck whose base is the lesser circle and height about a sixth of an inch. The upper half of each neck is filed square and a skrew hole is made in the middle of it. The round plates n, o are skrewed upon the surface of the pin board with flat headed skrews sunk below the surface of the plates; but the plate at p is first let into the pin board as deep almost as the plate is thick and then is skrewed down.

Three steel pins made to fit the round holes k, l, m in the 1st, 5th and 2^d sliders, already mentioned, are riveted to the far ends of three flat draught irons N, O, P , and each pin is kept firm to each iron plate by a shoulder below and a collar above.

Cut three slits in the fore board at r, s, t , directly opposite to the centers n, o, p , and having put the shank y of the straightest draught iron P thro' the slit t , and its steel pin into the column of holes in the sliders at m , and the large cylindrical hole P over the neck p which just fits it; by moving the shank sideways the 2^d slider which

has the round hole, will be moved alone by the steel pin. And to keep this slider from being stirred by a like motion of those above or below it, a brass washer, or circular springing plate, whose diameter it equal to that of the brass circle below, is fitted tight upon the square part of the brass neck and pressed down upon the draught iron by a steel skrew skrewed into the hole p in the middle of the neck.

The other two draught irons O, N are made crooked to go round about the end of the bridge and the extreme pins in the pin board, and are placed in like manner as before upon the brass circles o, n ; that upon o having another large hole wide enough to receive the skrew head at p and to give liberty to its own angular motion about the neck o . The hole at N in the third iron plate being put upon the neck n and over the two former plates, has two other holes wide enough to receive the skrew heads at o and p , and also to afford room for its own angular motion about the neck n , the planes of these two iron plates being set off upwards with cranked necks in order to move them above the other plates.

Allowing $\frac{1}{15}$ of an inch for the motion of any slider, or pen of a jack, the motion of any iron shank at its slit is to $\frac{1}{15}$ of an inch, in the given ratio of $p t$ to $p m$, which motion is therefore determined; and the breadth of any shank at the slit added to its motion there, gives the length of
the

the slit, which length, if experience shall require it, must either be augmented, or blocked up a little at either end or both, for adjusting the proper quantity of the slider's motion.

In the like vacant places at the base end of the sliders, three other columns of holes must be drilled thro' them all, and the holes in the 3^d, 6th and 4th sliders remaining round (*b*) all the rest must be lengthened on both sides, on purpose to draw any of those sliders separately (without moving the rest) by steel pins fixed as before in the ends of three other draught irons, of similar shapes to those at the treble end: the straightest being next the side board and sunk down a little, the 2^d going over it, and the 3^d over both. Note that the draught irons move under the strings of the harpsichord.

Fig. 70. Two sliders being saved, two sounds belonging to two extreme notes at each end of the two progressions by v^{ths}, are changed both together by the extreme knobs, rather than any two belonging to the intermediate notes. For as the extreme sounds *G^b* and *D^b*, or *B^{*}* and *F^{**}*, are seldom used than any of the means that are out of the vulgar scale (*i*), if either of their usual substitutes, *F^{*}* and *C^{*}*, or *C* and *G*, which are both excluded by the principals, *G^b* and *D^b*, or *B^{*}* and *F^{**}*, should chance to occur and interfere with them in the same piece of music, it follows that

(*b*) See those numbers in the lower line of Fig. 70.

(*i*) See the Table in Plate XIX.

that such accidents will happen seldomer in the former than in the latter case.

Upon communicating this method of changing the sounds of a harpsichord at pleasure, to two of the most ingenious and learned gentlemen in this University, the Reverend Mr. *Ludlam* and the Reverend Mr. *Michel*, they encouraged me to put it in practice upon a harpsichord made on purpose by Mr. *Kirkman*; and were so kind not only to direct and assist the workmen, but also to improve the usual method of drawing the sliders in the accurate steady manner above described; which answers the design so well, that a musician even while he is playing, can without interruption change any sound for another which he perceives is coming into use: which however is seldom required if the scale be properly adjusted before he begins to play.

It may not be amiss to observe that a careful workman might save some time and labour, if instead of widening each hole in the sliders separately, he should pierce out some long holes between those couples of narrow ones which move the jacks; leaving a few slender cross pieces here and there to add sufficient strength to the slider while he is working it.

SECTION IX.

Methods of Tuning an organ and other instruments.

PRACTICAL PRINCIPLES.

I. A consonance of any two musical sounds is imperfect if it beats or undulates, and is perfect if it neither beats nor undulates.

II. Any little alteration of the interval of a perfect consonance makes it beat or undulate, slower or quicker according as the alteration is smaller or greater.

III. If the interval of a perfect consonance be a little increased, the imperfect one is said to Beat Sharp; if a little diminished, to Beat Flat (*k*).

IV. An imperfect consonance will be discovered to beat sharp, if *a very small* diminution of its interval retards the beats; or to beat flat if any diminution accelerates them.

V. The harmony of a consonance is the finest and smoothest when it neither beats nor undulates, and grows gradually coarser and rougher while the beats are gradually accelerated by very small alterations of the interval.

VI. A small alteration is sooner perceived in the rate of beating than in the harmony of a consonance, and both must be attended to in tuning
an

(*k*) See Schol. 5. to Prop. xx. in the Appendix.

an instrument, especially the harpsichord, where the beats are weak and of short duration.

VII. If any imperfect consonance be sounded immediately after another, an attentive ear can determine very nearly whether they beat equally quick, or else which of them beats quicker, even without counting the beats made in a given time; especially upon the organ, where the beats are strong and durable at pleasure.

VIII. If several imperfect consonances of the same name, as v^{th} for instance (by which the whole scale is usually tuned) beat equally quick, they are not equally harmonious; to make them so, the higher in the scale ought to beat as much quicker than the lower as their bases vibrate quicker; that is, if a v^{th} be a tone higher than another, it should beat quicker in the ratio of 10 to 9 or of 9 to 8 nearly; if a iii^d higher, in the ratio of 5 to 4; if a v^{th} higher, of 3 to 2; if an $viii^{th}$ higher, of 2 to 1; &c.

IX. Pl.xx. Tab. iv. v. Beginning at any note as *G* of the undermost progression by tones, if two ascending v^{th} *G d, d a* and the descending $viii^{th}$ *a A* and the two next ascending v^{th} *A e, e b* be all made perfect, the vi^{th} *G e, d b* and the x^{th} *G b* will be found to beat sharp and so quick as to offend the ear by the coarseness of their harmony compared with that of the v^{th} . Which shews that in order to make all those concords and their complements to, and compounds with $viii^{th}$ more equally harmonious, the intervals
of

of the v^{th} s and consequently of the vi^{th} s and x^{th} s must be a little diminished in the following manner.

P R E C E P T S

for tuning an organ or harpsichord by estimation and judgment of the ear.

1. Pl. xx. Tab. v. Alter successively the trebles of the ascending v^{th} s $G d, d a$ till they beat flat, the lower v^{th} very slowly and the higher a little quicker; and the descending $viii^{\text{th}}$ $a A$ being made perfect, alter the trebles of the next ascending v^{th} s $A e, e b$ till they beat flat a *very little* quicker than the two former *respectively*.

Then if the x^{th} $G b$, between the base of the first and treble of the fourth v^{th} , beats sharp and about as quick as the v^{th} $G d$ to the same base, those 6 notes are properly tuned for the defective scale of Organs and Harpsichords in common use.

But if that x^{th} $G b$ beats sharp considerably quicker than the v^{th} $G d$, every one of those four v^{th} s $G d, d a, A e, e b$ must be made a *very little* flatter in order to beat a *very little* quicker than before.

On the contrary, if the x^{th} $G b$ beats sharp, considerably slower than the v^{th} $G d$, or not at all, or flat, those four v^{th} s must be made sharper, to beat flat a *very little* slower in the first case, and still slower in the second and third cases; till an equality

equality of beats of the said x^{th} and v^{th} to the same base G be nearly obtained.

In like manner, the descending $viii^{\text{th}}$ $b B$ being made perfect, let the two next ascending v^{ths} Bf^* , $f^* c^*$ be made to beat flat *a very little* quicker than the v^{ths} Ae , eb respectively, so as make the x^{th} Ac^* beat sharp about as quick as the v^{th} Ae to the same base.

Lastly make the iii^{d} eg^* beat as quick as the v^{th} eb to the same base: Because the iii^{d} beats just as quick as the x^{th} to the same base, of necessity.

If we had begun at the lowest note E^b , the whole scale might have been tuned by ascending two v^{ths} and descending an $viii^{\text{th}}$ alternately; but it is better to tune the lower part of it in going backwards from G , ascending by an $viii^{\text{th}}$ and descending by two v^{ths} alternately, as follows.

Let the ascending $viii^{\text{th}}$ $G g$ be made perfect, and by altering successively the bases of the descending v^{ths} gc , $c F$, make them beat flat *a very little* slower than the v^{ths} ad , $d G$ respectively, till the x^{th} Fa and the v^{th} Fc , to the common base F , beat equally quick; and thus those two v^{ths} are properly tuned, and so may the rest as the notes direct.

Then tune perfect $viii^{\text{ths}}$ to every one of those Notes.

2. Pl. xx. Tab. v. If the instrument has a changeable scale (*l*) having put out all the sounds by their stops but those of the common scale, the

(*l*) Sect. VIII. art. 11, 18, 19.

the method of tuning it is the same as before, observing only that every one of the four v^{th} s, as Gd , da , Ae , eb , must beat flat *a very little* quicker than before in the common defective scale, till every v^{th} and vi^{th} to the same base beat equally quick, the v^{th} s flat and the vi^{th} s sharp, as Gd and Ge , da and db , Ae and Af^* , &c. and these being so adjusted, every x^{th} and iii^d as Gb and GB , df^* , Ac^* , eg^* , will beat slowly flat as they ought to do.

Likewise in tuning backwards from G or g , the descending v^{th} s gc , cF must also beat flat *a very little* slower than a d , dG respectively, till every v^{th} and vi^{th} to the same base, as cg and ca , Fc and Fd , &c beat equally quick; and these being so adjusted the x^{th} and iii^d as Fa and FA , &c, will beat flat very slowly, as they ought. Then having tuned 8^{ths} to all the sounds of the common scale, change E^b , B^b , F , C , G , D and all their 8^{ths} into D^* , A^* , E^* , B^* , F^{**} , C^{**} and their 8^{ths} , and according to the notes in Tab. II.^d Pl. xx, proceed as before to tune the rest of the ascending v^{th} s G^* d^* , d^* a^* , &c.

Lastly change G^* , C^* , F^* , &c and their 8^{ths} into A^b , D^b , G^b &c and their 8^{ths} , and according to the notes in the same Table proceed as before to tune the rest of the descending v^{th} s e^b A^b , a^b d^b , d^b G^b , &c.

But if d^b be the extreme flat note in the instrument, alter the base of the v^{th} A^b e^b till it beats flat and just as fast as the vi^{th} A^b f beats sharp; likewise

likewise alter d^b till the v^{th} $d^b a^b$ beats flat and just as fast as the vi^{th} $d^b b^b$ beats sharp.

Then put out all the sounds but those of the common scale, to be changed again as occasions shall require; and the harmony of the whole will be much more delicate than usual in the opinion of such judges as can distinguish when harmony is fine and when it is coarse (*m*).

PROPOSITION XVIII.

To find the pitch of an organ.

THE FIRST METHOD.

By the following experiment made upon our organ at Trinity College, I found that the particles of air in the cylindrical pipe called *d*, or *de-la-sol-re* in the middle of the open diapason (*n*) made 262 complete vibrations, or returns to the places they went from, in one second of time. And this number of vibrations is what I call the Pitch of the Organ.

Having suspended a brass weight of seven pounds averdupois at one end of a copper wire, commonly used for some of the lowest notes of a harpsichord, I lapped the other end round a peg, taken from a violin, that turned stiffly in a hole made in the wainscot near the organ. Then by turning the peg to and fro I lengthened or shortened

(*m*) See Prop. xx. Schol. 1. art. 5.

(*n*) See the notation, Tab. iv. Plate xx.

ened the vibrating part of the wire, till it found a double octave below the sound of the pipe *d* abovementioned. Then having measured the length of the vibrating part of the wire, while stretched by the weight, from the loop below to the under side of the peg, and added to it the semidiameter of the peg, I cut the wire, first at the point of contact with the side of the peg, and then at the loop below; for I found no need of a bridge either above or below. And having repeated the experiment with another piece of wire taken from the same bunch, I found no sensible difference either in the length or weight of the vibrating part; the length being 35.55 inches and the weight 31 grains troy; and the seven pounds averdupois that stretched it, was equal to 49000 grains troy, allowing 7000 for each pound.

Hence by a Theorem hereafter demonstrated (*o*), the number of semivibrations, forwards and backwards together, made by the wire in one second was 131, and the number of such vibrations made by the air in the pipe *d*, two octaves higher, was 4×131 (*p*), and so the number of complete vibrations (*q*) was 2×131 or 262 Q. E. I.

Scholium.

I made that experiment in the month of September at a time when the thermometer stood at temperate or thereabouts.

N

But

- (*o*) Prop. xxiv. coroll. 1, 2. (*p*) Sect. 1. art. 3 and 7.
 (*q*) Sect. 1. art. 12.

But upon a cold day in November I found by a like experiment, that the same pipe gave but 254 complete vibrations in one second; so that the pitch of its sound was lower than in September by something more than $\frac{1}{4}$ of a mean tone.

And upon a pretty hot day in August I collected from another experiment, that the same pipe gave 268 complete vibrations in a second of time; which shews that its pitch was higher than in November by almost half a mean tone.

By some observations made upon the contraction and expansion of air, from its greatest degree of cold in our climate to its greatest degree of heat (*r*), compared with Sir *Isaac Newton's* theory of the velocity of sounds, I find also that the air in an organ pipe may vary the number of its vibrations made in a given time, in the ratio of 15 to 16; which answers to the major hemitone or about $\frac{7}{12}$ of the mean tone and agrees very well with the foregoing experiments.

Coroll. In order to know when the pitch of an organ varies, and when it returns to the same again, it is convenient to keep a thermometer constantly in the organ case.

(*r*) See Mr. *Cotes's* xvth Lecture upon Hydrostat. and Pneumat. towards the end.

THE

THE SECOND METHOD

of finding the pitch of an organ, or the number of vibrations made in a given time by any given note.

Let the notes *C, D, E, F, G, A, B, c, d, e, f, g, a, b, c'*, be in the middle of the scale, and from any Base note not higher than *D*, tune upwards three successive perfect v^{th} *DA, Ae, eb*, and downwards the perfect v^{th} *bd*; then having counted the number of beats made in any given time by the imperfect $viii^{th}$ *Dd*, the number of complete vibrations made in that time by its treble *d*, will be 81 times that number of beats.

For example suppose the $viii^{th}$ *Dd* be found to beat 65 times in 20 seconds; then 81×65 or 5265 is the number of complete vibrations made by the treble *d* in 20 seconds; and $\frac{5265}{20}$ or 263 is the number made in one second.

The time may be measured either by a watch that shews seconds, or a pendulum-clock, or a simple pendulum that vibrates forwards or backwards in one second, whose length from the point of suspension to the center of the bullet is 39 inches and one eighth. And the person that observes the measure of the time must give a stamp with his foot at the beginning and end of it, while another person counts the number of the beats made in that time; which number diminished

nished by one, is the number of the intervals between those successive beats and properly speaking is the number required. For greater accuracy the experiment should be repeated several times and a medium taken among the several results.

DEMONSTRATION.

For if the notes *D, E, F, &c.* stand for the times of the single vibrations of their sounds, we have $D : A :: 3 : 2$, $A : e :: 3 : 2$, $e : b :: 3 : 2$ and $b : d :: 3 : 5$; and by compounding those ratios we have $D : d :: 81 : 40$, which ratio being resolved into $81 : 80$ and $80 : 40$, shews that the $viii^{th}$ *Dd* is tempered sharp by a comma. Whence in cas. 1. Prop XI, putting $p = 1 = q$, $n = 1$, we have $\beta = \frac{2q^n M}{161p + q} = \frac{M}{81}$ and $81\beta = M$ the number of complete vibrations of the treble *d* of that $viii^{th}$.

By the first method *d* was found to vibrate 262 times in one second, which number being substituted for *M* shews that the $viii^{th}$ *Dd* in that organ made $\frac{262}{81}$ or $3\frac{11}{81}$ beats in one second, which are not too quick to be counted.

Coroll. 1. Hence the numbers of vibrations made in a given time by any sounds in the Diatonic System (*s*) are given in the given organ ;
being

(*s*) Sect. II^d. art. 1.

being reciprocally in the known ratios of their single vibrations.

Coroll. 2. By diminishing each of the ascending perfect $v^{\text{ths}} DA, Ae, eb$ by $\frac{1}{4}$ of a comma and increasing the descending $v^{\text{th}} bd$ by $\frac{1}{4}$ of a comma, the $viii^{\text{th}} Dd$ which was too sharp by a comma, becomes perfect: which is another proof of the vulgar temperament (*t*).

THE THIRD METHOD

of finding by experiment the number of vibrations N made by any given sound c of a given organ in a known time T.

In the same notation as before let ca be a sharp v^{th} making β beats in the time T ; make ad, dG, GC perfect v^{ths} , and Cc will be an imperfect $viii^{\text{th}}$; which if sharp and making b beats in the time T , will give $N = 81b + 16\beta$; or if flat, $N = 16\beta - 81b$.

For $c : a :: 5 + \frac{\beta}{N} : 3$ (*u*) and $a : d :: 2 : 3$ and $d : G :: 2 : 3$ and $G : C :: 2 : 3$. Whence by compounding all those ratios, $c : C :: 40 + \frac{8\beta}{N} : 81$ and this $viii^{\text{th}}$ being sharp and making

$$N \quad 3 \qquad \qquad \qquad b$$

(*t*) *Coroll. Prop. 11^d.*

(*u*) *Prop. XI. cor. 7. cas. 1.*

b beats in the time T , gives $C : c :: 2 : 1 - \frac{b}{N}$ (x). Wherefore $80 + \frac{16\beta}{N} = 81 - \frac{81b}{N}$ and $N = 81b + 16\beta$.

The VIIIth Cc cannot well come out flat nor can its beats be too slow and indistinct, unless those of the sharp VIth ca were too quick to be easily counted. This may be collected from *the second method*.

THE FOURTH METHOD

of finding the pitch of an organ, or N the number of vibrations of the sound c in a known time T.

The notation of the scale being still the same as in the *first method* and the octave Gg being perfect, let the vth cg , Gd , da , be made to beat flat 40, 30, 45 times respectively in a known time T ; then, if ca be a sharp VIth making β beats in the time T , we have $N = 3240 + 16\beta$. But if it be a flat VIth, $N = 3240 - 16\beta$.

For $c : g :: 3 - \frac{40}{N} : 2$ (y) and $g : G :: 1 : 2$ and $G : d :: 3 - \frac{30G}{Nc} : 2$ (z) and $d : a :: 3 - \frac{45d}{Nc} : 2$ (z). Whence by compounding all these ratios,

(x) Prop. XI. cor. 7. cas. 1.

(y) Prop. XI. cor. 7. cas. 2.

(z) Prop. XI. cor. 7. and coroll. 1. to the 2^d method.

ratios, $c : a :: 27 - \frac{1080}{N} : 16$; where if ca be a sharp $v^{1\text{th}}$ making β beats in the time T , $c : a :: 5 + \frac{\beta}{N} : 3$ (a). Whence $27 - \frac{1080}{N} : 16 :: 5 + \frac{\beta}{N} : 3$ and therefore $N = 3240 + 16\beta$. But if ca be a flat $v^{1\text{th}}$, $N = 3240 - 16\beta$.

If the resulting $v^{1\text{th}}$ ca beats sharp and too slowly and therefore too indistinctly, or else too quick to be easily counted; make the $v^{1\text{th}}$ cg beat slower or quicker respectively. Also if that $v^{1\text{th}}$ beats flat and consequently too slowly, make the $v^{1\text{th}}$ cg beat much quicker. The most convenient rate of beating is between 2 and 3 beats in a second *.

PROPOSITION XIX.

The pitch of an organ and the temperament of the $v^{1\text{th}}$ being given, to find the numbers of beats that every $v^{1\text{th}}$ will make in a given time.

The musical notes in Tab. 1 or 2, Plate xx, shew all the $v^{1\text{th}}$ s of different names in a complete scale of sounds; which $v^{1\text{th}}$ s by interposing $8^{1\text{th}}$ s are

N 4

placed

(a) Prop. xi. cor. 7. cas. 1.

* In July 1751 that excellent mathematician the Reverend and Learned Mr. *Tho. Bayes* F, R, S, was pleased to send me these two last methods, in return for a method of Tuning an Organ described in scholium 2 to Prop. xx, which I had sent him sometime before.

placed at such a pitch that the higher v^{th} s may not beat too quick to be counted, nor the lower too slow to be distinguished. By the given temperament of the v^{th} and the given pitch of the organ, the number β of the beats made in a given time by the v^{th} d may be found by Prop. XI, and also the constant ratio v to 1 of the times of the single vibrations of the base and treble of the v^{th} .

Then if a series of v^{th} s equally tempered ascend continually from d , as in Tab. 3, the numbers of their beats made in a given time, will continually increase in the ratio of 1 to v (b) and therefore will be β , βv , βv^2 , βv^3 , βv^4 , βv^5 , βv^6 , βv^7 , βv^8 , βv^9 .

Now as often as any of these v^{th} s are depressed by an octave, as in the upper half of the scale ascending from d in Tab. 1 or 2, so often must these beats be divided by 2 (b); which changes that series of beats into this, β , $\frac{\beta v}{2}$, $\frac{\beta v^2}{2}$,

$$\frac{\beta v^3}{4}, \frac{\beta v^4}{8}, \frac{\beta v^5}{8}, \frac{\beta v^6}{16}, \frac{\beta v^7}{16}, \frac{\beta v^8}{32}, \frac{\beta v^9}{32}.$$

In like manner, if the former series of v^{th} s be continued downwards from d , as in Tab. 3, the numbers of their beats made in a given time will continually decrease in the ratio of v to 1 or of

$$1 \text{ to } \frac{1}{v}, \text{ and therefore will be, } \frac{\beta}{v}, \frac{\beta}{v^2}, \frac{\beta}{v^3}, \frac{\beta}{v^4}, \frac{\beta}{v^5},$$

$$\frac{\beta}{v^6}, \frac{\beta}{v^7}, \frac{\beta}{v^8}, \frac{\beta}{v^9}, \frac{\beta}{v^{10}}.$$

Now

(b) Prop. XI. coroll. 2.

Now as often as these v^{th} s are raised by an octave, as in the lower half of the scale descending from d , in Tab. 1 or 2, so often must these beats be multiplied by 2; which produces this series of beats, $\frac{\beta}{v}, \frac{2\beta}{v^2}, \frac{2\beta}{v^3}, \frac{4\beta}{v^4}, \frac{8\beta}{v^5}, \frac{8\beta}{v^6}, \frac{16\beta}{v^7}, \frac{16\beta}{v^8}, \frac{32\beta}{v^9}, \frac{32\beta}{v^{10}}$.
Q. E. I.

Scholium.

For example, be it proposed to calculate the number of beats, in Tab. 1, Plate xx, which every v^{th} in the system of mean tones will make in 15 seconds of time.

Here the temperament of the v^{th} is $\frac{1}{4}$ of a comma (c), and supposing the interval of the perfect $v^{\text{th}} = \log. \frac{3}{2} = 0.1760913$, we have the comma $c = \log. \frac{81}{80} = 0.0053950$, and $\frac{1}{4}c = 0.0013488$, and the $v^{\text{th}} - \frac{1}{4}c = 0.1747425$, which is the logarithm of the number 1.4953 or $\frac{1.4953}{1}$, that is, of the ratio of 1.4953 to 1, which in the solution of the problem we represented by v to 1.

Now when the thermometer stood at temperate, the pitch of our organ at Trinity College, or the number of complete vibrations made in 1 second by the air in the pipe denoted by d in the middle of our table, was 262 (d).

Hence

(c) Prop. 11^d. cor.

(d) Prop. xviii.

Hence to find the number of beats made in 15 seconds by the v^{th} above d when tempered flat by $\frac{1}{4}$ comma, in prop. XI we have $m : n :: 3 : 2$ or $m = 3$, $n = 2$ and $\frac{q}{p}c = \frac{1}{4}c$, or $q = 1$, $p = 4$, and since the base d makes 262 complete vibrations in 1 second, in the given time of 15 seconds it will make 15×262 such vibrations = N , and the

Abacus 1.

O. 1747425	v	N° of beats.
1. 5629841	$\beta = 36, 558$	$\beta = 37$
1. 7377266	$\beta v = 54, 667$	$\frac{1}{2} \beta v = 27$
1. 9124691	$\beta v^2 = 81, 747$	$\frac{1}{2} \beta v^2 = 41$
2. 0872116	$\beta v^3 = 122, 24$	$\frac{1}{4} \beta v^3 = 31$
2. 2619541	$\beta v^4 = 182, 79$	$\frac{1}{8} \beta v^4 = 23$
2. 4366966	$\beta v^5 = 273, 34$	$\frac{1}{8} \beta v^5 = 34$
2. 6114391	$\beta v^6 = 408, 73$	$\frac{1}{16} \beta v^6 = 26$
2. 7861816	$\beta v^7 = 611, 20$	$\frac{1}{16} \beta v^7 = 38$
2. 9609241	$\beta v^8 = 913, 95$	$\frac{1}{32} \beta v^8 = 29$
3. 1356666	$\beta v^9 = 1366, 7$	$\frac{1}{32} \beta v^9 = 43$

the number of beats made in that time, by cas. 2,
 is $\frac{2qmN}{161p+q} = \frac{2 \times 3 \times 15 \times 262}{161 \times 4 + 1} = \frac{1572}{43} = \beta$, whose
 logarithm is 1.5629841; to which adding conti-
 nually the log. of v , we get the logarithms of
 βv ,

Abacus 2.

$\bar{1}. 8252575$	$\frac{1}{v}$	N ^o of beats
$1. 5629841$	$\beta = 36, 558$	$\beta = 37$
$1. 3882416$	$\frac{\beta}{v} = 24, 448$	$\frac{\beta}{v} = 24$
$1. 2134991$	$\frac{\beta}{v^2} = 16, 349$	$\frac{2\beta}{v^2} = 33$
$1. 0387566$	$\frac{\beta}{v^3} = 10, 933$	$\frac{2\beta}{v^3} = 22$
$0. 8640141$	$\frac{\beta}{v^4} = 7, 3116$	$\frac{4\beta}{v^4} = 29$
$0. 6892716$	$\frac{\beta}{v^5} = 4, 8896$	$\frac{8\beta}{v^5} = 39$
$0. 5145291$	$\frac{\beta}{v^6} = 3, 2699$	$\frac{8\beta}{v^6} = 26$
$0. 3397866$	$\frac{\beta}{v^7} = 2, 1867$	$\frac{16\beta}{v^7} = 35$
$0. 1650441$	$\frac{\beta}{v^8} = 1, 4623$	$\frac{16\beta}{v^8} = 23$
$\bar{1}. 9903016$	$\frac{\beta}{v^9} = 0, 9779$	$\frac{32\beta}{v^9} = 31$
$\bar{1}. 8155591$	$\frac{\beta}{v^{10}} = 0, 6540$	$\frac{32\beta}{v^{10}} = 21$

βv , βv^2 , βv^3 &c, as in the first Abacus, and thence the corresponding numbers, which divided by the proper powers of 2, as directed in the solution of the problem, give the ascending half of the set of beats opposite to the pitch 262 in Tab. 1.

The log. of v subtracted from 0 gives the log. of $\frac{1}{v}$, which log. continually added to the log. of β , gives the logarithms of $\frac{\beta}{v}$, $\frac{\beta}{v^2}$, $\frac{\beta}{v^3}$, &c as in the 2^d Abacus. And these logarithms give the numbers themselves, which multiplied by the proper powers of 2, as above directed, give the descending half of the same set of beats opposite to the pitch 262 in Tab. 1.

The superior sets of beats are designed for tuning the same or different organs, when their pitch is higher than this by 1, 2, 3 or 4 quarter-tones, as noted at the beginning of the table, and may be found by the continual addition of the logarithm of a quarter-tone to the logarithms in each Abacus; and the first inferior set of beats may be found by the subtraction of the log. of a quarter-tone from the said set of logarithms, or by the addition of its arithmetical complement: remembering to divide and multiply the corresponding numbers by the same powers of 2 as before in each Abacus.

And as the $\frac{1}{4}$ tone is $\frac{1}{8}$ of the III^d, its logarithm is $\frac{1}{8} \log. \frac{5}{4} = 0.0121138$, which continually

nually added to the logarithm of 262, gives the successive logarithms of the higher pitches 269, 277, &c, in the first column of the table, over against the corresponding superior sets of beats; and subtracted from the same logarithm of 262, gives the log. of the lowest pitch 255, overagainst the lowest set of beats.

From the given temperament of the system of equal harmony (*e*) the beats of all the v^{th} s may be calculated by the same method; and will be found as in Tab. 11^d. Plate xx.

Coroll. 1. Supposing the middlemost notes *d*, in the first and second tables, to be unisons, the numbers of the beats in a given time, of any two corresponding v^{th} s are very nearly in the given ratio of their temperaments $\frac{9}{36}c$ and $\frac{10}{36}c$ (*f*), or as 9 to 10. For the beats would be in that ratio if their several base notes were exactly unisons (*g*); and the difference of their pitches at the distance of the tenth v^{th} from the middle note *d*, is but ten times the difference of the temperaments, or $\frac{5}{18}c$; which produces the difference of but 1 beat in 290 in the extreme v^{th} s in the tables, and less in the rest in proportion to their distances from the note *d* (*h*).

Coroll. 2. In the system of equal harmony the ratio of the numbers of beats of the 111^d and v^{th} to

(*e*) Prop. xvi, Schol. 2. art. 10.

(*f*) Prop. 2. cor. and prop. xvi. schol. 2. art. 10.

(*g*) Prop. xi. coroll. 4. and schol. 1.

(*h*) Cor. 4. lemm. to prop. ix. and cor. 2. prop. xi.

to the same base, is 2 to 3 in a given time; as being compounded of the ratio of their temperaments $\frac{2}{18}$ and $\frac{5}{18} c$, and of the major terms of their perfect ratios 5 : 4 and 3 : 2 (*i*).

Coroll. 3. For the same reason the beats of the v^{th} and vi^{th} to the same base are isochronous in the system of equal harmony, whereas in that of mean tones they are in the ratio of 3 to 5 in a given time.

PROPOSITION XX.

To tune any given organ by a given table of beats.

Having found the pitch of the organ by any of the methods in Prop. xviii, look for the nearest to it in the first column of Table I or II, Plate xx, and overagainst it is the *Proper Set of Beats* for tuning the given organ. If the weather be considerably hotter or colder at the time of tuning than it was when the pitch was found, allowance must be made in the number of vibrations denoting the pitch, by the schol. to prop. xviii.

Then flatten the treble *a* of the perfect v^{th} above *d* (*k*) more or less, till the number of its beats, made in 15 seconds (*l*), agrees with the tabular

(*i*) Coroll. 3. prop. xi and schol. i.

(*k*) See the notation tab. iv. plate xx.

(*l*) To be measured as directed in the Second Method in prop. xviii. But in this case count one beat more than the tabular number, as properly signifying the number of intervals between the successive beats.

tabular number placed over that v^{th} in the proper set.

From the treble of that v^{th} *da* tune downwards the octave *aA*, so as to be quite free from beats, and repeat the like operation upon the next ascending v^{th} *Ae*, and the like again upon the next till you have tuned all the sharp notes in the scale of your organ.

Then going backwards from *d*, sharpen the base *G* of the perfect v^{th} below *d* more or less till the number of its beats, made in 15 seconds, agrees with the tabular number placed over this v^{th} in the proper set.

From the base of the v^{th} *dG*, tune upwards the octave *Gg* and repeat the like operation upon the next descending v^{th} *gc*, and the like again upon the next, till you have tuned all the flat notes in the scale of your organ.

This being done, let all the other sounds be made octaves to these, and the scale will be exactly tuned according to the temperament in the given table; that is, all the v^{ths} will be equally tempered, and consequently equally harmonious (*m*), and so will all the vi^{ths} , and every other set of concords of the same name, which answers the design of tuning by a table of beats.

If you chuse to tune the organ according to the *Hugenian* system, the set of beats in Tab. 1, next below that which answers to the pitch, found by any of the foregoing methods, will serve your purpose.

For

(*m*) Prop. XII coroll.

For the *Hugenian* v^{th} , having its temperament $-\frac{1}{4}c + \frac{1}{110}c$ smaller than $-\frac{1}{4}c$, in the ratio of 53 to 55 (n), beats slower than the tabular v^{th} in that proportion, which is but very little slower than it would do, if its pitch were depressed by $\frac{1}{4}$ of a mean tone. Q. E. F.

Scholium I.

1. Since our organ at Trinity College was new voiced, and by altering the disposition of the keys was depressed a tone lower, and thereby reduced to the Roman pitch, as I judge by its agreement with that of the pitch pipes made about the year 1720; by the help of such a pipe one may know by how many quarter-tones the pitch of any other organ is higher than that of ours, and thus (without any of the methods in Prop. xviii) determine the proper set of beats for tuning it (o).

2. At a time when the thermometer stood at Temperate, as it did also when the pitch of our organ was found to be 262, I assisted at the tuning of the v^{ths} of the open Diapason by the set of beats opposite to that pitch in Tab. 1, and upon examining the iii^{ds} and x^{ths} I found them all perfect: a manifest proof of the theory of beats and of the certainty of success in tuning by it.

3. At that time the whole organ was tuned to the open diapason, and is now universally allowed to

(n) Prop. xvii. Schol. Tab. 2.

(o) See column 2, Tab. 1, 2, plate xx.

to be much more harmonious than before, when the major thirds were much sharper than perfect ones; and its harmony, I doubt not, is still improveable by making them flatter than perfect, according to the system of equal harmony. But at that time I had not finished the calculation of it, and to repeat the tuning of the organ over again would be troublesome and improper at the present season, when cold and damp weather is coming on very fast.

4. For the properest times for tuning the Diapason of an organ seem to be from the latter end of August to the middle of October, when the air being dry, temperate and quiet, will keep nearer to the same degree of elasticity for a given time. Because a very small alteration in the warmth of moist air will suddenly and sensibly alter its elastic force and thereby the pitch of the pipes before the whole stop can be accurately tuned.

For that reason constant care must be taken not to heat the pipes by touching them oftener than is needful; nor to stay too long at a time in the organ case; nor to tune early in the morning, but rather towards the evening, when the air is drier and its declining warmth is kept at a stay by the warmth of the persons about the organ.

But these and the like cautions may sooner be learned by a little practice than by any description, and if not altogether necessary, will however contribute to the accuracy of tuning by so nice a method which is plainly capable of any

O

desired

desired degree of exactness provided the blast of the bellows be uniform.

5. After tuning an organ according to any new system whatever, we must be cautious of judging too hastily of it. Some musicians here who had constantly been used to major thirds and consequently major sixths tuned very sharp, could not well relish the finer harmony of perfect thirds and better sixths in the organ newly tuned, till after a little use they became better satisfied with it, and after a longer use they could not bear the coarse harmony of other organs tuned in the usual manner.

It is therefore necessary to have equal experience in different objects of sense, in order to judge impartially, which of the two is more grateful than the other, as is evident in almost every thing to which we are more or less habituated.

6. If a machine were contrived, as it easily might, to beat like a clock or watch, any given number of times in 15 seconds, between 20 and 56 or thereabouts; by setting it to beat according to any given number in the table for tuning an organ, and by comparing its beats with those of the corresponding v^{th} , the ear would determine immediately and exactly enough, whether they were isochronous or not; and thus a harpsichord might be tuned almost to the same exactness as an organ; and the tuning of an organ might be performed much quicker by the help of such a machine than by counting the beats as above.

In

In the following method after 3 or 4 fifths are tuned by a little table of beats, the organ itself does the office of such a machine in all the rest.

Scholium 2.

To tune any given organ by isochronous beats of different concords.

1. The sounds in the middle of the scale being called *C D E F G A B c d e f g a b c'*, make the VIII^{th} *Gg* quite perfect, and let the v^{th} *cg*, *Gd*, *da* be made to beat flat 38, 28, 42 times respectively in 15 seconds of time.

N ^o of beats of the v th <i>ca</i>	The system of Equal Harmony. Tab. 1.			A proper system for defective scales. Tab. 2.			
	The numbers of beats of the v th s.						
	<i>cg</i>	<i>Gd</i>	<i>da</i>	<i>cg</i>	<i>Gd</i>	<i>da</i>	<i>Ae</i>
56	41.	31—	46—	32+	24—	36—	27—
50	40.	30—	45—	31+	23+	35—	26+
44	39.	29+	44—	31—	23—	34+	25+
38	38.	28+	42+	30—	22+	33+	25—
32	37.	28—	41+	29—	22—	32+	24+
26	36.	27—	40+	28+	21+	31+	24—
20	35.	26+	39+	27+	20+	31—	23—

Then having counted the number of beats made in 15 seconds by the resulting v^{th} *ca*, look for it or the nearest number to it in the column of the beats of that v^{th} placed before the tables. The

numbers opposite to it in each table are the Proper Set of Beats, which the said v^{th} ought to make in the given organ, when tuned according to the system mentioned in the title of each table.

2. For example, suppose the resulting vi^{th} should beat 48 times in 15 seconds, the nearest number to it in the first column is 50 and opposite to it in Tab. 1, are 40, 30, 45, the proper set of beats of those v^{th} cg , Gd , da for causing the v^{th} and vi^{th} , cg and ca , to beat equally quick as they ought to do in the system of equal harmony (p).

Likewise in Tab. 2, opposite to the said number 50 are 31, 23, 35, 26, the proper set of beats of the v^{th} cg , Gd , da , Ae , for causing the v^{th} and iii^{d} , cg , ce , to beat equally quick, which property makes a very proper system for the defective scales of organs and harpsichords in present use.

3. Alter then the trebles of the v^{th} cg , Gd , da till they beat flat 40, 30, 45 times respectively in 15 seconds, remembering by the way to make gG a perfect $viii^{\text{th}}$. Then upon sounding the v^{th} and vi^{th} cg , ca , immediately after each other, the ear will judge near enough whether they beat equally quick as they ought.

4. But if greater certainty be desired, count the beats of the vi^{th} ca in 15 seconds and if their number be 40 as that of the v^{th} cg was, or differs from it by no more than 3, over or under, you have found the proper set; but if the beats of the

vi^{th}

(p) Prop. XIX. schol. coroll. 3.

v^{th} differ from 40 by more than 3 and less than 9, over or under, the next higher or lower set respectively, is the proper set. Because an alteration of six beats of the v^{th} answers in time to an alteration of but one beat of the v^{th} to the same base, as appears by the differences of the numbers in the first and second columns. For which reason, if in any experiment the beats of the v^{th} *ca* should come out exactly in the middle between any two numbers in the 1st column, you may take either of the sets opposite to them for the proper set; the error from the truth being but half a beat, that is, half the interval of the successive beats of the v^{th} *cg*, which half cannot easily be measured. Yet the choice might be determined by altering the pitch of the sound *c* a very little and repeating the experiment.

5. The proper set for a given organ being once found, the 1st experiment need never be repeated afterwards. For whatever be the proper set in temperate weather, the next above it will be the proper set in hot, and the next below it in cold weather (*q*). At most, the set so determined need only be verified as in Article 3 or 4 whenever the organ wants to be retuned.

6. After the v^{th} *cg*, *Gd*, *da* had been made to beat 38, 28, 42 times in 15 seconds in the 1st experiment, if the beats of the v^{th} *ca* had come out too quick, or too slow and indistinct to

O 3

be

(*q*) Prop. XVIII. scholium.

be easily counted, (which however cannot happen unless the pitch of the organ be immoderately higher or lower than usual,) in the first case use 13 seconds, and in the second 17 instead of 15; and upon repeating the experiment the beats of the v^{th} *ca* will come out within the limits 20 and 56, of the numbers in the first column and point to the proper set.

7. Lastly, if the beats of any v^{th} cannot soon be adjusted to the tabular number, which sometimes happens, and that number has the sign + after it, the excess of a beat may be dispensed with as being less erroneous in that case than the defect of a beat; and on the contrary, if the tabular number has the sign - after it.

8. Pl. xxv. Fig. 65. Now if you chuse to tune the organ to the system of equal harmony, which being the most harmonious is the properest for a changeable scale (*r*), in ascending from the sounds, *c*, *G*, *d*, *a* tuned by the proper set of beats, make the v^{th} *Ge* beat sharp and just as quick as the given v^{th} *Gd*, and do the like for every v^{th} in the order annexed.

	v	vi	viii
	<i>Gd</i> ,	<i>Ge</i>	
	<i>da</i> ,	<i>db</i> & tuning <i>aA</i>	
	<i>Ae</i> ,	<i>Af</i> *	
	<i>eb</i> ,	<i>ec</i> * & tuning <i>bB</i>	
	<i>Bf</i> *	<i>Bg</i> *	
	&c,	&c	&c.

Likewise in descending from the said given sounds

(*r*) Sect. viii art. 11th.

founds *c*, *G*, *d*, *a*, alter the common base *F* of the v^{th} and vi^{th} ,

<i>Fc</i> and <i>Fd</i> , till	v	vi	$viii$
they beat equally	<i>Fc</i> ,	<i>Fd</i> & tuning	<i>Ff</i>
quick, the v^{th} flat,	<i>B^bf</i> ,	<i>B^bg</i>	
and the vi^{th} sharp;	<i>E^bB^b</i> ,	<i>E^bc</i> & tuning	<i>E^bc^b</i>
and repeat the like	<i>A^bE^b</i> ,	<i>A^bf</i>	
practice in the or-	&c,	&c	&c.

der annexed.

If this scale ascend so high as to cause a v^{th} and vi^{th} , as *eb* and *ec** for instance, to beat too quick, tune downwards the two $viii^{\text{th}}$ *eE* and *bB* and leaving out the uppermost row of the sharp notes, proceed with the lower rows. And do the contrary in the descending part of the scale, leaving out the undermost flat notes where you find they beat too slow and descending by the higher notes.

9. But till Instruments are made with a changeable scale, it is more proper to tune the defective scale in present use by making every v^{th} and iii^{d} , to the same base, beat equally quick, the former flat and the latter sharp.

Make the v^{th} *cg*, *Gd*, *da*, *Ae* beat flat 31, 23, 35, 26, times respectively, this being the proper set found by Art. 1, 2, for the given organ, and the iii^{d} *ce* will beat sharp equally with the v^{th} *cg*.

Plate xxv. Fig. 65. Then in ascending from the sounds *c, g, G, d, a, A, e* so tuned, make the next III^d

<i>GB</i> beat sharp and	v	III	VIII
equally with the given	<i>Gd,</i>	<i>GB</i> and tuning <i>Bb</i>	
en v th <i>Gd,</i> and do the	<i>da,</i>	<i>df*</i>	
like for the rest in the	<i>Ae,</i>	<i>Ac*</i>	
order annexed.	<i>eb,</i>	<i>eg*</i>	

Likewise in descending from the same given sounds alter the base *F* of the III^d *FA*

<i>Fc,</i>	<i>FA,</i> & tuning <i>Ff</i>
<i>Bbf,</i>	<i>Bbd</i>
<i>EbBb,</i>	<i>EbG.</i>

till it beats sharp and equally with the given vth *Fc* to the same base, and do the like for the rest in the order annexed.

DEMONSTRATION.

Pl. xxiv. Fig. *A*, which is described in Prop. III and cor. 1. 2. The beats of any given vth and vith to the same base will be isochronous, as they ought to be in the system of equal harmony, when their temperaments are as 5 to 3 (*t*). Whence by the coroll. to prop. iv, *Gr* is $= \frac{5}{18} c$ and is the flat temperament of all the v^{ths} (*u*) and *As* $= \frac{3}{18} c$ is the sharp temperament of the resulting vi^{ths}.

In Tab. 1. column 1 the ascending numbers 35, 36, 37, 38, &c are assumed for the beats to be made in a given time by the given vth *cg* in organs

(*t*) Prop. xi. coroll. 3 and schol. 1.

(*u*) Prop. xii. coroll.

organs of different pitches (x) and from these and the given temperament $\frac{5}{18}c$, the beats of the other v^{th} Gd , da in col. 2. 3 of that table are computed by the method in Prop. XIX.

When the v^{th} and v_1^{th} cg and ca , have any other temperaments, as $G\rho$ and $A\sigma$, let a and b be the respective numbers of their beats made in any known time, and β their common number made in that time when their temperaments were $Gr = \frac{5}{18}c$ and $As = \frac{3}{18}c$ as before. Then supposing the difference $rg = \frac{d}{18}c$, by the similar triangles $Or\rho$, $Os\sigma$, we have $s\sigma = \frac{3d}{18}c$. And since the numbers of beats made in equal times by a given consonance differently tempered are proportional to its temperaments (y), we have $\beta : a :: Gr : G\rho :: 5 : 5 - d$, whence $d = \frac{5\beta - 5a}{\beta}$; likewise $\beta : b :: As : A\sigma :: 3 : 3 + 3d$, whence again $d = \frac{b - 3}{\beta}$. Therefore $5\beta - 5a = b - \beta$ and $\beta = a + \frac{b - a}{6}$.

In the method above for finding the proper set of beats we assumed $a = 38$ and in the example we supposed $b = 48$, whence $\beta = 38 + \frac{48 - 38}{6} = 39\frac{2}{3}$, the nearest to which in column 1. tab. I being 40, gives the proper set, 40, 30, 45. After

(x) Prop. XI. cor. 2.

(y) Prop. XI. cor. 4. and schol. I.

After this manner I found the proper set for making the v^{th} and v_1^{th} , cg , ca , beat equally quick in our organ at Trinity College, to be 36, 27—, 40 +, in a warm season, after the pitch of it had been depressed a tone lower, to the Roman pitch, only by changing the places of the keys. Consequently the proper set for the former pitch would have been 40, 30—, 45—, because the ratio of 36 to 40 or 9 to 10 belongs to the tone nearly. For this reason in the method above I chose to begin the experiment with the intermediate set 38. 28 +, 42 +, and adapted to it the column of beats of the v_1^{th} ca in the following manner.

When $a = 38$, $\beta = 38 + \frac{1}{6} \overline{b-38}$, whence it appears if $b = 38$, that $\beta = 38$; if b exceeds 38, that β exceeds 38 by $\frac{1}{6}$ of that excess; if b be deficient from 38, that β is deficient from 38 by $\frac{1}{6}$ of the defect. So that the common difference of the values of β or beats of the v^{th} cg being unity in col. 1. Tab. I, the common difference of the correspondent values of b or beats of the v_1^{th} ca must be six, as in the column prefixed to the Table.

If $b = 0$, that is, if the v_1^{th} ca should come out perfect, we have $\beta = 38 - 6 \frac{1}{3} = 31 \frac{2}{3}$, which in col. 1. Tab. I continued downwards would give the pitch and the proper set for an organ about a tone lower than ours, tho' lower than ordinary;

ordinary; because the ratio of 36 to $31\frac{2}{3}$ or 32 , is 9 to 8 belonging to the tone major. Therefore in the experiment above for finding the proper set, the v^{th} ca cannot beat flat upon any organ now in use: and if its sharp beats come out too slow or too quick, a remedy has been given above in art. 6. The reason of it is this; if either of the temperaments $G\rho$, $A\sigma$, of the v^{th} and vi^{th} be increased, the other will decrease, and accordingly the beats of the former consonance will be accelerated while those of the latter are retarded. So that when the beats of the vi^{th} come out too quick to be easily counted, they shew that those of the v^{th} were too slow at first and therefore must be accelerated, or the same number of them must be made in a less time: and on the contrary, when the beats of the vi^{th} come out too slow.

In Tab. 2, where the major III^d , in order to lessen the diesis (z), is designed to beat sharp and as quick as the v^{th} beats flat, the ratio of the temperaments $G\rho$, $E\tau$, of the v^{th} and III^d must be 5 to 3 (a), and the temperer $O\rho\sigma\tau$ must be within the angle EOG . Hence $G\rho = \frac{5}{23}c(b)$.

But in Tab. 1 we had $Gr = \frac{5}{18}c$, and so the ratio of the beats made in a given time by the correspondent

(z) Sect. VIII. art. 2. note u .

(a) Prop. XI. cor. 3. schol. 1.

(b) Prop. v. cor. cas. 2, where $\frac{r}{t}$ may have any value, because the second condition of the proposition is not here required.

spondent v^{th} in Tab. 1 and 2, is Gr to Gg or 23 to 18 (c).

In like manner a table of beats for any other system might be computed and subjoined to Tab 1, in which the proper set may be found as before in the 1st experiment. Another system might be tuned by isochronous beats of the 111^d and v_1^{th} , but it differs so little from that of equal harmony that the mention of it is sufficient.

Corollary. Hence we have the number of vibrations made in a given time by any given sound of a given organ. For the number of complete vibrations made in a given time by the sound c is the product of this constant number $96,7\frac{2}{3}$ multiplied by the number of beats made in that time by the v^{th} cg when it beats equally with the v_1^{th} ca , to be found by the method above described.

Thus if that number of beats be 36 in 15 seconds, in this time the sound c makes $36 \times 96,766$ & $c = 3484$ complete vibrations, that is $232\frac{1}{5}$ vibrations in one second; and therefore the sound d , which is almost a mean tone higher than c , makes 260 such vibrations in one second, which agrees with the experiment made with the brass wire in prop. xviii.

For the temperament of the v^{th} cg , when it beats flat and equally with the v_1^{th} ca , was found to be $\frac{5}{18}c, = \frac{q}{p}c$ in prop. xi, where in cas. 2, $\beta =$

29mN

(c) Prop. xi. coroll. 4.

$\frac{2qmN}{161p+q}$, or $N = \frac{161p+q}{2qm} \beta = \frac{161 \times 18 + 5}{10 \times 3} \beta =$
 $96, 7\frac{2}{3} \times \beta$, the number of vibrations of the
 found c .

Scholium 3.

If we could measure any given part of a second of time more readily and exactly by any other means than by the beats themselves, a single set of beats for a given system, as 38, 28, 42, &c would alone be sufficient for tuning any given organ according to the given system.

For, by the method above, having found β the proper number of beats which the v^{th} cg ought to make in a given time as 15 seconds, the time t , in which it will make the number 38 in the given set, is to 15 seconds, as 38 is to β ; and that time so determined is the Proper Time in which all the other numbers of beats in the given set ought to be made by the other v^{th} in the said organ. But unless that time t , which will generally contain some fraction of a second, could be readily and accurately measured, this method will with equal expedition be less accurate, or with equal accuracy will be less expeditious than the former.

For if instead of the mixt number t we use the nearest whole number of seconds for the proper time, the limit of the error will be half a second; whereas in using the Proper Set for any given time, the error is but half the interval of the successive beats of the v^{th} cg , which is two
 or

or three times smaller than half a second, because the number of its beats in col. 1. Tab. 1, is always between two and three times greater than 15, the number of seconds in which the beats are made. And the larger error cannot easily be reduced to an equality with the smaller, unless by a set of beats whose numbers are between 2 and 3 times larger than those in the Tables, which would proportionally increase the time and trouble of counting them. For instance, instead of counting 38, 28, 42 beats in 15 seconds, we must count 96, 72, 107, in 38 seconds. Because 15, 38, 96 are continual proportionals nearly.

As the known method of tuning an instrument by the help of a monochord is easier than any other to less skilful ears, and pretty exact too if the *apparatus* to the monochord be well contrived, it may not be amiss to shew the manner of dividing it according to any proposed temperament of the scale.

PROPOSITION XXI.

To find the parts of a given monochord, whose vibrations shall give all the sounds in an octave of any proposed tempered system.

Let the system of equal harmony be proposed, and let the several parts of the monochord be measured from either end of it, and be to the whole,

whole, in the ratios of the several numbers in the 3^d column of the following table, to 100000; I say the vibrations of the parts so found, and of the whole, will give all the sounds in an octave of the proposed system, as denoted in the first column of the table. Q. E. I.

For in the scholium to prop. xvii we had $2T = 0.09631.05650$ and $2L = 0.06025.35832$; whence we have

$$\begin{aligned} T &= 0.04815.52825 \\ L &= 0.03012.67916 \\ \hline T-L &= l = 0.01802.84909 \\ L-l &= d = 0.01209.83007 \end{aligned}$$

From these logarithms of the tone, limma major and minor and the diesis, and from the logarithm 4.69897.00043 of the number 50000, the uppermost in the table, all the logarithms below it will be found by the following additions: where the musical notes in column 1 are supposed also to represent the logarithms over against them, till you come down to C, which comes out 5.00000.00000 and shews that the

$$\begin{array}{lll} c + d = B^*, & c + l = c^b, & c + L = B \\ B + l = B^b, & B + L = A^*, & B + T = A \\ A + l = A^b, & A + L = G^*, & A + T = G \\ G + l = G^b, & G + L = F^*, & G + T = F \\ F + d = E^*, & F + l = F^b, & F + L = E \\ \&c & \&c & \&c \end{array}$$

logarithms

logarithms of the principal notes B, A, G, F, E, D are right; and those of the secondary notes will be right too, if the operations in the addition be right.

The corresponding numbers in column 3, which may be found by the tables of logarithms, shew the required parts of the monochord; as a very little reflection will satisfy any one that understands the common properties of logarithms, and attends to the intervals of an octave in Fig. 49 described in Sect. VIII, but not divided as there into 50 equal parts, which is only an approximation to the system proposed. Q. E. D.

Scholium.

The numbers in the 4th and 5th columns of the table shew the parts of a monochord, whose vibrations will give the sounds of the opposite notes in the system of mean tones and that of Mr. *Huygens*, who has shewn how to find the last column of numbers in his *Harmonic Cycle*. And as all the measures in the 3 systems may be taken and marked upon the sounding board of the same monochord, the different effects of those systems upon the ear, may be easily tried and compared together, provided the tone of the monochord be good and the divisions accurate, and the moveable bridge does not strain it in one place more than in another.

SECTION

The division of a Monochord.

1	2	3	4	5
<i>c</i>	4. 69897. 00043	50000	50000	50000
<i>B*</i>	4. 71106. 83050	51412	51200	51131
<i>c^b</i>	4. 71699. 84952	52119	52245	52278
<i>B</i>	4. 72909. 67959	53592	53499	53469
<i>B^b</i>	4. 74712. 52868	55863	55902	55914
<i>A*</i>	4. 75922. 35875	57441	57243	57179
<i>A</i>	4. 77725. 20784	59876	59814	59794
<i>A^b</i>	4. 79527. 05693	62412	62500	62528
<i>G*</i>	4. 80737. 88700	64177	64000	63942
<i>G</i>	4. 82540. 73609	66897	66874	66866
<i>G^b</i>	4. 84343. 58518	69733	69877	69924
<i>F*</i>	4. 85553. 41525	71702	71554	71506
<i>F</i>	4. 87356. 26434	74742	74767	74776
<i>E*</i>	4. 88566. 09441	76853	76562	76467
<i>F^b</i>	4. 89159. 11343	77910	78125	78196
<i>E</i>	4. 90368. 94350	80110	80000	79964
<i>E^b</i>	4. 92171. 79259	83506	83593	83621
<i>D*</i>	4. 93381. 62266	85865	85599	85512
<i>D</i>	4. 95184. 47175	89504	89443	89422
<i>D^b</i>	4. 96987. 32084	93298	93459	93512
<i>C*</i>	4. 98197. 15091	95934	95702	95627
<i>C</i>	5. 00000. 00000	100000	100000	100000
	System of	equal harmony	mean tones	Mr. Huy- gens.

SECTION X.

Of occasional temperaments used in concerts well performed upon perfect instruments.

By a perfect instrument I mean a voice, violin or violoncello, &c, with which a good musician can perfectly express any sound which his ear requires.

PROPOSITION XXII.

The several Parts of a concert well performed upon perfect instruments, do not move exactly by the given intervals of any one system whatever, but only pretty nearly, and so as to make perfect harmony as near as possible.

For instance, if the base be supposed to move by the best system of perfect intervals (*d*), the other part or parts cannot constantly move by it too, without making some of the concords imperfect by a comma (*e*), which would grievously offend the musicians (*f*). Consequently if they are pleased, those intervals are occasionally tempered

(*d*) See Sect. II. art. I.

(*e*) Sect. IV. Prop. I. and coroll.

(*f*) Sect. IV. art. 9.

pered by the upper part or parts, which therefore do not move by the same intervals which the base is supposed to move by.

Likewise if the base be supposed to move by the system of mean tones and limmas (*g*), the other part or parts cannot constantly do so too, without making about two thirds of all the concords imperfect by a quarter of a comma (*b*). But whenever concords are held out by good musicians, they seem to me to be always perfect. And if so, the upper part or parts cannot move by the system of mean tones, which the base is supposed to move by. And the argument is the same if the base be supposed to move by any other system of tempered intervals: and that it cannot constantly move by perfect ones, I shall shew in the next scholium.

What has been said of perfect and imperfect concords, is applicable to discords too, a good ear being critical in both. Now the reason why the best musicians acquire a habit of making perfect harmony, as near as possible, is plainly this. When the harmony is made perfect they are pleased and satisfied, though the several parts do not move by perfect intervals. For the passing from one sound to the next, whether by a perfect or an imperfect interval, being nearly instantaneous, cannot much offend the musician. But the succeeding consonance is long enough held out to give him pleasure or pain according as he makes it perfect or imperfect. Q. E. D.

Coroll.

(*g*) Prop. II.

(*b*) Prop. III. coroll. 3.

Coroll. 1. Cæteris paribus the harmony will be the same whatever be the system which the base moves by; but the sum of the occasional temperaments will be the least possible, if it moves by the system of mean tones and limmas (*i*), and but very little bigger, if it moves by the scale of equal harmony (*k*).

Coroll. 2. The proposition holds true though one of the instruments be imperfect, as when the thorough base is played upon an organ or harpsichord: because the performers of the upper parts are more attentive to make perfect harmony with the base notes, than with the chords to them. Consequently those parts do not move by the tempered system of the thorough base.

Coroll. 3. Cæteris paribus the same piece of music well performed upon perfect instruments, is more agreeable than it would be if it were as well performed upon imperfect ones, as an organ, &c.

For nothing gives greater offence to the hearer, though ignorant of the cause of it, than those rapid rattling beats of high and loud sounds, which make imperfect consonances with one another. And yet a few slow beats, like the slow undulations of a close shake now and then introduced, are far from being disagreeable.

Coroll. 4. Therefore the harmony of a concert will be smoother and distincter, and generally

P 2

rally

(*i*) Prop. III. coroll. 10.

(*k*) Schol. to prop. xx. art. 6. in the Appendix. and prop. 41. coroll. 4.

rally more pleasing, for taking the chords of the thorough base as near as can be to the base notes, and no more of them than are necessary, and these few upon the softer and simpler stops of an organ.

Because the beats will then be fewer, slower and softer, and so the voices and other instruments will appear to greater advantage.

Coroll. 5. It appears also from the reasons above, that no voice part ought ever to be played on the organ, unless to assist an imperfect singer, and keep him from making worse concords with the base and other parts than the organ it self does.

Scholium.

Mr. *Huygens* observed long ago, that no voice or perfect instrument can always proceed by perfect intervals, without erring from the pitch at first assumed (*l*). But as this would offend the ear of the musician, he naturally avoids it by his memory of the pitch, and by tempering the

(*l*) Aio itaque, si quis canat deinceps sonos, quos Musici notant literis C, F, D, G, C, per intervalla consona, omnino perfecta, alternis voce ascendens descendensque; jam posteriorem hunc sonum C, toto commate, quod vocant, inferiorem fore C priore, unde cani coepit. Quia nempe ex rationibus intervallorum istorum perfectis, quæ sunt 4 ad 3, 5 ad 6, 4 ad 3, 2 ad 3, componitur ratio 160 ad 162, hoc est 80 ad 81, quæ est commatis. Ut proinde, si novies idem cantus repetatur, jam propemodum tono majore, cujus ratio 8 ad 9, descendisse vocem, tonoque excidisse oportet. *Cosmotheoros* lib. I. pag. 77.

the intervals of the intermediate sounds, so as to return to it again (*m*).

This is also confirmed by what we are told of a monk (*n*), who found, by subtracting all the ascents of the voice in a certain chant from all its descents, that the latter exceeded the former by two commas: so that if the ascents and descents were constantly made by perfect intervals, and the chant were repeated but four or five times, the final sound, which in that chant should be the same as the initial, would fall about a whole tone below it. But finding that the voices in his choir did not vary from the pitch assumed, he concluded that the musical ratios, whereby he measured those successive ascents and descents, were erroneous. But if he had known Mr. *Huygens's* remark, it would have solved his difficulty.

This was not the first time that the truth of those musical ratios had been called in question. For *Galileo* observed that the reason commonly

P 3 al-

(*m*) Hoc verò nequaquam patitur aurium sensus, sed toni ab initio sumpti meminit, eodemque revertitur. Itaque cogimur occulto quodam temperamento uti, intervallaque illa canere imperfecta; ex quo multo minor oritur offensio. Atque hujusmodi moderamine ferè ubique cantus indiget; uti colligendis rationibus, quemadmodum hic fecimus, facile cognoscitur. *ibid.*

(*n*) Methode generale pour former les Systême temperés de musique. *Mem. de l'Acad. des Scienc. Ann. 1707. pag. 263. 8vo.*

alleged for it, appeared to him insufficient (*o*). At last indeed he hit upon a couple of experiments which gave him satisfaction (*p*), but a scientific proof was still wanting till Dr. *Taylor* published his theory of the vibrating motion of a musical chord (*q*), which has since been cultivated by several able mathematicians (*r*), and being the principal foundation of Harmonics, deserves to be further considered in the next section.

SECTION XI.

Of the vibrating motion of a musical chord.

PROPOSITION XXIII.

When a musical chord vibrates freely, the force which urges any small arch of it towards the center of its curvature,

(*o*) Stetti lungo tempo perplesso intorno à queste forme delle consonanze, non mi parendo che la ragione, che communemente se n'adduce da gli autori, che fin quì hanno scritto dottamente della musica, fosse concludente à bastanza. Dicono essi &c. *Discorsi attenenti alla Meccanica, Dialogo 1°*, towards the end.

(*p*) Ibidem.

(*q*) Methodus incrementorum, prop. 22, 23, and Philos. Transf. N^o 337, IV. or Abridg. by Jones Vol. 4. p. 391.

(*r*) Commentarii Acad. Petropol. Tom. III.

Comment on the Principia Vol. 2. pag. 347.

Mr. Maclaurin's Fluxions, art. 929.

ture, is to the tension of the chord in the ultimate ratio of the length of that arch when infinitely diminished, to the radius of its curvature.

I suppose the chord to be uniform, and very slender, or rather to be a mathematical line, flexible by the least force and elastic; and its tension or quantity of elastic force to be measured by a weight, which if hung to one end of it, would distend it to the same length which it has when it vibrates freely by the force of its elasticity.

Pl. XXI. Fig. 50. Let ACB represent such a chord fixed at the points A and B , CD any arch of it, CE and DE tangents at C and D ; in either of which as EC produced if need be, take EF equal to ED , and draw FG perpendicular to FE , and DG to DE , and joining DF produce GE towards H .

Then imagine the chord to keep its curvature while a force applied at E or H draws the tangents EC , ED and these the points of contact C , D , so as to keep them in *æquilibrio*. And since the elastic forces at C and D are each equal to the force of tension, the direction of the third force at E will bisect the angle CED under the other two directions, and consequently will coincide with the line GEH , agreeably to the construction of the equal triangles EDG , EFG .

Hence the three forces at H , C and D , which would keep the point E at rest, are proportional to the sides DF , FG , GD of the isoscelar triangle DFG , to which their several directions EH , EC , ED are perpendicular; because that triangle is similar to any other, as EDI , whose sides are either parallel to, or in part coincident with those directions, and therefore proportional to the forces acting in them, by the known theorem in Mechanics (*s*).

Now suppose CL and DM to be the *radii* of the curvatures of the chord at the points C , D , and the curve LM to be the *locus* of all the centers of the curvatures at every point of the arch CD . Then conceiving the point D to move up to C , and consequently M and G up to L , the limit of the variable ratio DF to FG , of the said forces, will be that of the evanescent arch CD to CL the *radius* of its curvature. And a force constantly equal and opposite to the former of the two, is that which urges the vanishing arch CD in the direction EG , which ultimately coincides with CL ; and the latter was the force of tension. Q. E. D.

Coroll. When a musical chord vibrates freely, the forces which accelerate its smallest equal arches, are constantly proportional to their curvatures very nearly, provided the latitude of the vibrations be very small in proportion to the length of the chord.

For

(*s*) See Theorem 33 of Keill's Physics.

For the force of its tension being then very nearly invariable, the forces which accelerate its smallest equal arches are very nearly in the inverse ratio of the *radii* of their curvatures (*t*), which is the same as the direct ratio of the curvatures themselves.

D E F I N I T I O N
OF THE HARMONICAL CURVE.

Fig. 51. *Let C be the common center of any two circles DF, EG, and CDE, CFG any two semidiameters, and of either of the included arches as DF, let FH be the sine, in which produced both ways, let the lines HI and HK be severally equal to the other arch EG; then while the semidiameter CFG moves round the center C and carries with it the line IFHK, parallel to it self and constantly equal to twice the arch EG, the extremities I, K will describe a curve whose vertex is D and axis DC, and whose base ACB is equal to the semicircumference of the circle EG.*

(*t*) By the present Proposition.

Coroll.

Coroll. 1. Pl. xxii. Fig. 52. Drawing FL perpendicular to the base ACL , a line KP perpendicular to the curve at K , will be parallel to EL .

For drawing KN perpendicular to the base, let the radius CFG go forwards a little into the place Cfg , and carry the line $KHFI$ into the place kbf_i , cutting KN in O and FL in r . Then since $HK = EG$ by the definition, and also $bk = Eg$, their difference Ok is $= Gg$. Now by the similar triangles CLF and Fr_f , CFf and CGg , OKk and NPK , we have $CL : CF :: Fr : Ff$, and $CF : CG :: Ff : Gg$, and *ex æquo* $CL : CG$ or $CE :: (Fr : Gg :: OK : Ok ::) NP : NK$. Consequently the right angled Triangles CLE , NPK are equiangular, and the perpendicular KPM is parallel to the line EL .

Coroll. 2. At any point K the radius of curvature $KM : LE :: LE \text{ quad.} : KN \times CE$.

For drawing fl parallel to FL ; another line kM , perpendicular to the curve at k , will be parallel to El , by *coroll. 1*; consequently if the arch Kk be infinitely diminished, either of the coinciding perpendiculars KM , kM will be the radius of the curvature at K .

In the line El take $Es = EL$ and joining Ls , the triangles LEs and KMk , CEL or CEl and sLl , are ultimately equiangular.

Now KN or $FL : LC :: fr : rF$ or KO , and $LC : LE :: PN : PK :: KO : Kk$, and *ex æquo*, $KN : LE :: fr : Kk$;

But

But $CE : LE :: sL : Ll$ or fr , therefore *componendo*, $KN \times CE : LE \text{ quad.} :: (sL : Kk ::) LE : KM$.

Coroll. 3. Hence if the ratio of the circles CEG , CDF be vastly great, the curvature at any point K will be extremely small, and its radius $KM : CE :: CE : KN$ very nearly; because the lines LE and CE will be very nearly equal.

Coroll. 4. Upon the same supposition, the very small curvatures at any points D , K are very nearly in the ratio of their distances DC , KN from the base AB .

For when CE and consequently AB is given, the curvature at K , being reciprocally as its radius KM , is directly as KN by *coroll. 3*.

Coroll. 5. Fig. 53. While the greater circle remains let the lesser be diminished, and the curve $AKDB$ will be changed into another $A\kappa\delta B$ of the same *species*, and every ordinate to the common base will be diminished in the same ratio, that is, $NK : N\kappa :: CD : C\delta$.

Fig. 52. For while any arch EG equal to HK or CN is given in magnitude, let the other radius CD or CF be diminished, and because the triangle CFH retains its *species*, the line CH or NK is diminished in the same ratio with CF or CD .

Coroll. 6. Fig. 53. When the axes CD , $C\delta$ of two curves are very small in comparison to their common base AB , the curvatures at the tops of
any

any two coincident ordinates NK , $N\kappa$, are in the ratio of the ordinates.

For if $\kappa\mu$ be the radius of curvature at κ , by coroll. 3 we have $KM \times KN = CE \text{ quad.} = \kappa\mu \times \kappa N$; whence $\kappa N : KN :: KM : \kappa\mu$, that is, as the curvature at κ to the curvature at K .

Coroll. 7. Hence, supposing the curve $AKDB$ to have the elasticity and tension of a musical chord, it will vibrate to and fro in curves very nearly of the same species with the given curve $AKDB$, provided none of the vibrations be too large.

For let the first effort of the tension reduce that curve into some other, as $A\kappa\delta B$, in the first moment of time; and since the ordinates DC , KN are in proportion as the curvatures at D and K by coroll. 4, and those curvatures as the accelerating force at D and K (u), acting in the directions DC , KM or KN very nearly, and these forces as the velocities generated by them in that time, and the velocities as the nascent spaces $D\delta$, $K\kappa$; *alternando*, we have $DC : D\delta :: KN : K\kappa$ and *dividendo*, $DC : \delta C :: KN : \kappa N$. Consequently by coroll. 5 the curve $A\kappa\delta B$ is very nearly of the same species with $AKDB$. And in the next moment it will be changed into another of the same species, and so on, till every point of the chord be reduced to the base AB at the same instant. And by the motion here acquired it will be carried towards the opposite side of the base, till by the opposition of the tension, it shall lose

all

(*u*) Coroll. prop. XXIII.

all its motion by the same degrees, and in the same curves, by which it was acquired; and thus the chord will continually vibrate in curves of the same species as the first, neglecting the small difference in the directions KN , KM , and the resistance of the air.

Coroll. 8. The small vibrations of a given musical chord are isochronous.

For if the chord at the limit of its vibration assumes the form of the harmonical curve, it will vibrate to and fro in curves of that species by coroll. 7, and its several particles, being accelerated by forces constantly proportional to their distances from the base AB (x), will describe those unequal distances in equal times, like a pendulum moving in a cycloid.

If the chord at the limit of its vibration assumes any other form, it will cut an harmonical curve, equal in length to it, in one or more points, as A , K , L , B in Fig. 54; and the intercepted parts of the chord will be more or less incurvated towards AB than the corresponding parts of the curve, according as they fall without or within them; and will accordingly be accelerated by greater or smaller forces than those of the corresponding parts of the curve (y). Therefore, supposing the chord and curve to differ in nothing but their curvatures, the difference of the curvatures of the corresponding parts will be continually diminished by the difference of
their

(x) Cor. prop. xxiii and cor. 6. defin. curve.

(y) Cor. prop. xxiii.

their forces, till the parts coincide either before, or when they arrive at the base AB . And thus the times of the several vibrations of the chord will be the same as those of the curve, and therefore equal to one another.

Coroll. 9. The Figure contained under the harmonical curve and its base, is of the same species as the Figure of Sines.

Fig. 52. For supposing the circle DFQ to grow bigger till it becomes equal to EGR , the figure $AKDB$ will become a figure of sines. Because any ordinate KN to the absciss AN or arch GR , being constantly equal to FL , will then be equal to the sine of the arch GR ; and thus every ordinate as KN is increased in the given ratio of CF to CG , or CD to CE . And on the contrary the several ordinates in the said figure of sines diminished in that constant ratio of CE to CD , are the ordinates in the figure $AKDB$ of the harmonical curve.

PROPOSITION XXIV.

The vibrations of a musical chord stretched by a weight, are isochronous to those of a pendulum, whose length is to the length of the chord, in a compound ratio of the weight of the chord to the weight that stretches it, and of the duplicate ratio of the diameter of a circle to its circumference.

Fig.

Pl. XXIII. Fig. 54. If P be the weight that stretches the chord ADB , and DCM be the radius of its curvature at the vertex D , the force that urges any small particle Dd towards C is $= \frac{Dd}{DM} \times P$ by prop. XXIII.

And since Dd vibrates like a pendulum (z), if it were suspended by a string $OP = DC$ in a cycloid $QPR = 2DC$ or DCF , and were urged at the highest points Q, R by a force acting downwards like that of gravity, but equal to the said force $\frac{Dd}{DM} \times P$, which urges Dd at the limits D, F of its vibrations; the times of those oscillations and of these vibrations would be equal to one another. Because the forces being also equal at all other equal distances of the particle from P and from C , would impel it through equal parts of the equal lines QP, DC in equal times.

Again putting p for the weight of the chord ADB or ACB , the weight of its particle Dd is $= \frac{Dd}{AB} \times p$.

Hence if another string L be to the string OP or DC , as this latter weight $\frac{Dd}{AB} \times p$ is to the former $\frac{Dd}{DM} \times P$, equivalent to the force at D ; and the particle Dd be again suspended by the string L in another cycloid of the length $2L$; since

(z) Coroll. 8. def. of the curve,

since at the highest points of this cycloid the particle is urged downwards by the whole force $\frac{Dd}{AB} \times p$ of its own gravity, its oscillations will be isochronous to those of the former pendulum (a). Because we took their lengths in the ratio of the forces that act upon them at the highest points of the cycloids, that is, $L : DC :: p \times DM : P \times AB$; which two ratios compounded with DC to AB , give $L : AB :: p \times DM \times DC$ or $p \times CEq$ (b) : $P \times ABq$, which was to be proved. For CEq is to ABq in the duplicate ratio of the diameter to the circumference, by the definition of the curve; and we shewed above that every particle of the curve vibrates in the same time with the middlemost. Q. E. D.

Coroll. 1. The time of one semivibration, forwards or backwards, of the chord AB measured by inches and decimals, is $\frac{113}{355} \sqrt{\frac{p}{P} \times \frac{AB}{39.126}}$ and its reciprocal is the number of such vibrations made in one second.

For the length of a pendulum that vibrates forwards or backwards in one second, is 39.126 inches in the latitude of London, and the diameter is to the circumference of a circle as 113 to 355 very nearly, and the times of the vibrations of pendulums are in the subduplicate ratio of their lengths. Whence putting t for that of the

(a) Coroll. Theor. 4 De Motu Pend. in Mr. Cotes's Harmon. Mensurarum.

(b) Coroll. 3. defin.

the pendulum L , we have $t'' : 1'' :: \sqrt{L} = \frac{113}{355} \sqrt{\frac{p}{P} \times AB} : \sqrt{39.126}$, and $t'' = \frac{113}{355} \sqrt{\frac{p}{P} \times \frac{AB}{39.126}}$, and $\frac{1}{t} = \frac{355}{113} \sqrt{\frac{P}{p} \times \frac{39.126}{AB}}$, the number of semivibrations made in one second.

Coroll. 2. Supposing the last number to be n , we have the logarithm of $n^2 = \log. \frac{P}{p \times AB} + 2, 58676.52698$, which gives n very expeditiously.

For the logarithm of $\left(\frac{355}{113}\right)^2 \times 39.126 = 2, 58676.52698$.

Coroll. 3. If the lengths and tensions of two chords be equal, the times of their single vibrations are in the subduplicate ratio of their weights, by coroll. 1.

Coroll. 4. If their lengths and weights be equal, the times of their single vibrations are reciprocally in the subduplicate ratio of their tensions, by coroll. 1.

Coroll. 5. If their tensions be in the ratio of their weights, the times of their single vibrations are in the subduplicate ratio of their lengths, by coroll. 1.

Coroll. 6. The weights of cylindrical chords are in a compound ratio of their specific gravities, lengths and squares of their diameters, that is, p is as $s \times AB \times d^2$; whence t is as $AB \times d \sqrt{\frac{s}{p}}$, by coroll. 1.

Q

Coroll.

Coroll. 7. Hence, if the tensions and diameters of homogeneous chords be equal, the times of their single vibrations are in the ratio of their lengths.

Coroll. 8. If the tensions and lengths of homogeneous chords be equal, the times of their single vibrations are in the ratio of their diameters.

Coroll. 9. If the tensions of similar chords be as their specific gravities, the times of their single vibrations are in the duplicate ratio of their lengths or of their diameters (*c*).

Scholium.

1. Hence we may find the number of vibrations made in a given time by any musical sound, by comparing it with the sound of a given chord stretched by a given weight.

For example in the experiment abovementioned (*d*) I found the length of the vibrating chord $AB = 35.55$ inches and its weight $p = 31$ grains troy: And the sound of it, when stretched by the weight $P = 7$ pounds averdupois = 49000 grains troy, was two octaves below the sound of the pipe *d* there mentioned. Hence by coroll. 2, we have $n = 131.04$, the number of semivibrations made in one second by the wire AB ,

(*c*) See *Galileo's* experiments on chords. Dialogo 1^o attenente alla Meccanica, towards the end.

(*d*) Prop. XVIII.

AB , and $4n = 524.16$, the number of semivibrations made by $\frac{1}{4}AB$, by coroll. 7, or by the pipe d ; which is double the number 262 of its whole vibrations.

Before this experiment was made the orifice of the pipe was cut perfectly circular, and then the length of the cylindrical part was exactly 21.6 inches, and its diameter 1.9, which I mention because the experiment, being accurately made, is of use upon other occasions.

2. When the thermometer is at Temperate, the latitude of a pulse of the sound of that pipe is to the length of the pipe, almost as $2\frac{1}{2}$ to 1, by prop. L. Lib. II. Princip. Philos.

Q 2

AD-

ADVERTISEMENT.

THOUGH the theory of imperfect consonances has been demonstrated pretty clearly, I think, in the sixth Section, yet as I had considered some parts of it in different lights and searched a little further into some others for my own diversion, I thought it not amiss to print my papers in the form of the following Additions; that if the reader should desire any further information, he may have recourse to them whenever he pleases.

THE CONTENTS.

A scholium to prop. viii.

An illustration of prop. x, with a scholium confirming the theory of the beats of imperfect consonances.

Another demonstration of scholium 5. prop. xi, (concerning the analogy between audible and visible undulations) and of prop. vii.

Another demonstration of prop. xiii and its third corollary, with an illustration of the first, and a scholium or two confirming the theory of the harmony of imperfect consonances, and shewing the absolute times and numbers of their vibrations, short cycles and dislocations of their pulses, contained in the periods between their beats.

Schol. 4 to prop. xx, containing tables and observations on the numbers of beats of the concords in the principal systems.

Schol.

Schol. 5 to prop. xx, shewing the methods of altering the pitch of an organ pipe in order to tune it.

Scholium to Prop. viii.

Fig. 63. **W**HEN different multiples as $3AB$ and $2ab$ of the vibrations AB , ab of imperfect unisons, are the single vibrations AD , ac of an imperfect consonance, the multipliers 3 and 2 are in the ratio of the single vibrations $3AB$ and $2AB$, or $3ab$ and $2ab$ of the perfect consonance, and therefore should be irreducible to smaller numbers. The different multiples of the vibrations of imperfect unisons are therefore supposed in the proposition to be the least in the same ratio.

Pl. xxv. Fig. 63, 64. But if different multiples of the vibrations AB , ab , as $6AB$ and $4ab$, whose multipliers 6 and 4 are reducible by a common divisor, be the single vibrations of an imperfect consonance, (as they may by intermitting 6—1 pulses of AB and 4—1 of ab , so as to leave single pulses at first and between every intermission,) the period of the imperfections of this consonance will not be equal to that of the imperfect unisons AB , ab , but multiple of it by 2, the greatest common divisor of the multipliers 6 and 4.

For those multiple vibrations $6AB$ and $4ab$ are the same as $3 \times 2AB$ and $2 \times 2ab$, or $3AC$ and $2ac$, in which the equimultiples $2AB$ and $2ab$, or AC and ac in fig. 64, are the single vibrations of other imperfect unisons, resulting

Q₃

from

from an intermission of every second pulse of AB and ab in fig. 63; and the period of their imperfections is equal to that of $3AC$ and $2ac$, or AG and ae by this VIIIth proposition, and is the same multiple of the period of AB and ab , as AC is of AB , or ac of ab by the coroll. 4 to this proposition, that is by 2, the greatest common divisor of the multipliers 6 and 4.

An illustration of Prop. x.

THUS in Fig. 23, Pl. XI, after taking away 9 short cycles from each end of the cycle AU of imperfect unisons, there remains $kKLm$, part of two more; and in Fig. 25, after taking away 3 short cycles from each end of the period AX , there remains dDe , part of another; and in Fig. 27, after taking away 4 short cycles of imperfect octaves from each end of the period Aw , there remains $iILm$, part of 2 more; and lastly in Fig. 34, Pl. XII, after taking away 2 short cycles of imperfect v^{ths} from each end of the cycle AZ of the imperfect unisons, there remains $nN\mathcal{Q}r$, part of another: which though not situated exactly in the middle of AZ , by reason of the part $\Delta \epsilon Z$ of another short cycle, containing equimultiples of AB and ab , is comparatively very near it when the number of short cycles in the period is very large as usual; in which case the beats will be made very nearly in the middle of every period.

Scholium.

Scholium.

It is very unreasonable to suppose with Mr. *Sauveur* that the beats are made by the united force of the coincident pulses of imperfect unisons (*e*).

For while the imperfect unisons are made to approach gradually to perfection, experience shews that they always beat slower and slower (*f*) and by theory (*g*) the periods of their pulses grow longer and longer. Therefore in consequence of that gentleman's hypothesis, the unisons should also beat at the ends of the periods where the pulses do not coincide: Because it is very improbable that the cycle of unisons, supposing it simple at first, while it lengthens gradually, will not sometimes be changed into periods as well as into other simple cycles.

Nor can it be allowed that the unisons will beat only at the ends of their complex cycles. For according as the numeral terms expressing the ratios of the single vibrations of the several successive unisons, happen to be reducible or not reducible or to be irrational, the cycles of the pulses will sometimes be shortened, sometimes lengthened again, sometimes invariable and sometimes impossible, as shall be explained by and by; which accidents disagree with the constant gradual retardation of the beats in the present case.

Q₄

If

(*e*) Prop. xi. schol. 3.

(*f*) See Phænomena of beats placed before prop. x.

(*g*) Cor. 5. lemma to prop. ix.

If it be said that the pulses next to the periodical points fall so close to one another, as to affect the ear in the same manner as if they were quite coincident; it may be so, and most probably is so. And then it will follow that the harmony of the short cycles terminated by such close pulses, will there be much the same as that of perfect unisons; at least it will certainly be better about the periodical points and coincident pulses than any where else in the periods. But the sound of a beat has no harmony in it; on the contrary it rather resembles the common sound of a beat or stroke upon any gross, irregular body: And this sound results from pulses of air which rebounding from different parts of the body, disposed to vibrate in different times, will strike the ear one after another at irregular intervals, like the pulses in the middle between the periodical points of imperfect unisons. Therefore these are the only pulses in each period, which can excite the sensation of the beats of imperfect unisons. And the like argument is applicable to any other imperfect consonance by prop. VIII.

Pl. XXIV. Fig. 55. As to those uncertain lengths abovementioned of the simple and complex cycles of the pulses of imperfect unisons, while their interval is continually diminished or increased; let one of the sounds be fixt and the time of its single vibration be represented by any given line *V* and those of the variable sound by the successive lines *A*, *B*, *C*, &c, all which lines may constitute any increasing or decreasing progression; and

and supposing n to represent any large given number, let $A : V :: n : a$, $B : V :: n : b$, $C : V :: n : c$, &c.

Then will the cycles of the pulses of V and A , V and B , V and C , &c, be $nV = aA$, $nV = bB$, $nV = cC$, &c, provided every one of the numbers a , b , c , &c, be integers and primes with respect to the assumed number n . In which case the several cycles are equal to one another and to nV .

But if the terms of all or any of those ratios have a common divisor, the corresponding cycles will be shortened in proportion as the greatest common divisors are larger; and therefore their lengths cannot increase or decrease successively in regular order while the successive intervals of the unisons continually decrease or increase, unless the greatest common divisors decrease or increase in regular order too; which can happen but very rarely.

And when the terms of the ratios of any of the vibrations happen to be incommensurable, a second coincidence of their pulses will be impossible: because no multiple of one vibration can be equal to any multiple of the other.

But in all cases whatever, the periods of the pulses of V and A , V and B , V and C , &c, which are $\frac{nV}{n-a}$, $\frac{nV}{n-b}$, $\frac{nV}{n-c}$, &c (*b*), will decrease continually in the same proportions with the fractions

(*b*) Def. III. sect. VI.

tions $\frac{n}{n-a}$, $\frac{n}{n-b}$, $\frac{n}{n-c}$, whose magnitudes can never be altered by any common divisors of their terms, whether integers fractions or furds.

*Another demonstration of scholium 5.
Prop. xi, and of Prop. vii.*

THE breadth of the apparent Undulations of the lights and shades seen at a distance upon two rows of parallel objects, may be also found by the following construction.

Pl. xxiv. Fig. 56. Let a plane passing through a distant eye at z , cut the axes of the parallel objects at right angles in the points a, b, c , &c, α, β, γ , &c, which are supposed equidistant in both the parallel lines abc , $\alpha\beta\gamma$. From any object in one of these lines to any successive objects in the other, draw the lines αa , αb , αc , &c, and the lines $z\psi V$, $zx X$, $zy Y$, &c, drawn parallel to them, will intercept the equal breadths of the apparent undulations.

Because while the eye is gradually directed from the middle of any of the breadths VX , XY , &c, towards either of its extremities, the objects will appear closer together in couples, in proportion to their smaller distances from the next extremity, which was shewn in this scholium to be the cause of the undulations.

For the lines $d\delta$, $e\epsilon$, $f\zeta$, $g\eta$, &c, being parallel to $a\alpha$, are parallel to $V\psi$ by construction; and the lines $i\theta$, $k\iota$, $l\kappa$, $m\lambda$, &c, being parallel to $b\alpha$,

$b\alpha$, are parallel to Xx ; and so on. Let lines drawn from z through the objects δ , ε , ζ , &c, of one row, cut the line of the other in D , E , F , &c. Then because the rows are parallel, the ratio of $D\delta$ to δz , $E\varepsilon$ to εz , $F\zeta$ to ζz , &c, is the same as Vv to vz , or Xx to xz , &c (i). Whence also, because of the parallels between the rows, we have

$$\left. \begin{array}{l} Dd : dV :: D\delta : \delta z \\ Ee : eV :: E\varepsilon : \varepsilon z \\ Ff : fV :: F\zeta : \zeta z \\ Gg : gV :: G\eta : \eta z \\ Hi : iX :: H\theta : \theta z \\ Ik : kX :: I\iota : \iota z \\ Kl : lX :: K\kappa : \kappa z \\ Lm : mX :: L\lambda : \lambda z \\ \&c. \qquad \qquad \&c. \end{array} \right\} :: Vv : vz;$$

That is, all those ratios are equal, and, alternately, the lesser apparent intervals Dd , Ee , Ff , Gg , are proportional to their distance dV , eV , fV , gV , from the next extremity V of the breadth VX ; and also Hi , Ik , Kl , Lm , proportional to iX , kX , lX , mX , their distances from the next extremity X of the same breadth VX . And the breadths VX , XY , &c, are equal, because ab , bc , &c are so, and the triangles Vzx and $a\alpha b$, XzY and $b\alpha c$, &c, are similar by construction. Q. E. D.

Coroll. 1. The projections DE , EF , &c, of the equal intervals $\delta\varepsilon$, $\varepsilon\zeta$, &c, are to these intervals
in

(i) 2 VI. Euclid.

in the constant ratio of Dz to δz , or Ez to εz , or Vz to vz , and consequently are equal to one another. Therefore supposing the lines DE and $\delta\varepsilon$ or de to represent the times of the single vibrations of imperfect unisons, the periods of the nearest approaches of their pulses D, E , &c. d, e , &c, are VX, XY , &c; And in going from their extremities V, X to the middle, the alternate lesser intervals between the successive pulses, are proportional to their distances from the next extremity, as we shewed just now: which is another proof of prop. VII.

Coroll. 2. If the eye be moved in a line parallel to the rows, the breadths of the apparent undulations will be constantly the same, and if it be moved uniformly in any other right line, their breadths will vary uniformly, and be constantly proportional to the distance of the eye from the rows. Because the triangles VzX, VzY , &c, are constantly similar to $a\alpha b, a\alpha c$, &c. And this conclusion seems to agree with what I have transiently observed of these undulations.

But it is easy to collect from the construction of the figure, and the different ratios of zV to zv expressed by numbers, that the intervals between the apparent conjunctions of the objects will increase and decrease very irregularly; and that no conjunctions can happen except when the eye arrives at certain points of its course, and none at all, mathematically speaking, when its distances from the two rows, measured upon any right line, happen to be incommensurable.

Which

Which conclusions being contrary to the continual appearance of the undulations to the eye in all places, and to the regular increase or decrease of their breadth, shew, that their breadth is not equal to the interval between the apparent conjunctions, no more than the interval between the beats of imperfect unisons is equal to the interval between their coincident pulses.

L E M M A.

In any period between the successive beats of an imperfect consonance, any given number of short cycles next to one side of the least dislocation of the pulses, is more harmonious, and the same number of them next to the other side is less harmonious than the same number of them next to either side of the coincident pulses: and these degrees of harmony differ more in those periods where the two least dislocations differ less, and most of all in the periods where these dislocations are equal when possible.

Pl. xxv. Fig. 59. Let AB and ab represent the times of the single vibrations of imperfect unisons, A and a their coincident pulses, $B, C, D, \&c, b, c, d, \&c$, their successive pulses on each side of A, a ; Rr their least dislocation in any given period, and consequently the nearest to the periodical point z , which is here placed under A , for the convenience of seeing at one view,

view, the short cycles next to both sides of Rr and Aa .

First I say, the short cycles RS , ST , &c, which include z , are more harmonious, and RQ , QP , &c, less harmonious than AB , BC , &c, the numbers of them being the same: and that the degrees of their harmony differ more in the periods where the two least dislocations Rr , sS differ less, and most of all where $Rr = sS$, when possible (k).

$$\begin{array}{l} \text{For } bB = (AB - Ab = RS - rs =) Rr + sS \text{ (l).} \\ \text{And } cC = (AC - Ac = RT - rt =) Rr + tT. \\ \qquad \qquad \qquad \text{\&c.} \qquad \qquad \qquad \text{\&c.} \end{array}$$

Hence the successive dislocations sS , tT , &c, are respectively smaller than bB , cC , &c, by Rr , as appears also by their smaller distances from z (m). But on the other side of Rr , the dislocations Qq , Pp , &c, are respectively greater than bB , cC , &c, by the same Rr , for the like reasons.

Now the short cycle RS which includes z , is more harmonious than AB next to A . For though the dislocations $Rr + sS$ are $= bB$, yet those parts of bB , as being smaller than bB , will give less offence to the ear than the whole: the whole may be perceived and give some offence even when one or both its parts are imperceptible. And for the same reasons the short cycle RS will be still more harmonious than AB in other periods

(k) See prop. VII. coroll. 2.

(l) See prop. VII. coroll. 1.

(m) Prop. VII.

periods where Rr, sS are less unequal, and the most harmonious where they are equal when possible (n): their sum being every where the same.

The next short cycle ST is also more harmonious than BC ; the dislocations sS, tT being respectively smaller than bB, cC . Therefore the short cycles RS, ST , taken together, are more harmonious than AB, BC taken together; and still more harmonious in other periods where sS, tT are smaller, till sS be equal to Rr .

But on the other side of Rr and Aa , the short cycle RQ is less harmonious than AB , the dislocations Qq, Rr being larger than Bb and o . The next short cycle QP is also less harmonious than BC ; the dislocations Pp, Qq being respectively larger than Cc, Bb . Therefore RQ, QP together are less harmonious than AB, BC together; and still less harmonious in other periods where Rr, Qq, Pp are larger, till Rr be equal to sS . And the same is evident in any larger equal numbers of short cycles throughout the period between the successive beats.

Secondly, any imperfect unisons will be changed into imperfect octaves whose single vibrations are AC and Ab , or Ac and AB , by conceiving every second pulse of the series A, B, C , &c, or a, b, c , &c, to be intermitted, which would depress one of the unisons an octave lower.

Now if that intermission should take away the alternate pulses S, U , &c, or s, u , &c, the short
cycles

cycles of the octaves, next to one side of Rr , will be RT , TW , &c, and on the other, RP , PN , &c: I say the former as including z are more, and the latter less harmonious than AC , CE , &c, the numbers of them being equal.

For we had $Rr + tT = cC$, consequently the short cycle RT is more harmonious than AC , for the same reason as in unisons, and because the intermediate dislocations sS , bB are vanished, one of their constituent pulses in each being taken away. And RT is still more harmonious than AC in other periods where Rr and tT are less unequal.

The next short cycle TW is also more harmonious than CE , the dislocations tT , wW being respectively smaller than cC , eE , as in unisons; and is still more harmonious in other periods where tT , wW are smaller, that is where tT and Rr are less unequal.

But on the other side of Rr and Aa , the short cycle RP is less harmonious than AC , and PN than CE , the dislocations Rr , Pp being respectively bigger than o and Cc ; and Pp , Nn bigger than Cc , Ee , respectively: and is still less harmonious in other periods where Rr , Pp , Nn are larger, that is where Rr , tT are less unequal.

Therefore the short cycles RT , TW , &c are more, and RP , PN , &c are less harmonious than AC , CE , &c.

Likewise if that alternate intermission should take away the pulses R , T , W , &c, or r , t , w , &c, then the least dislocation is sS , and the short cycles

cycles SQ , QO , &c, as including z , will be more, and SU , UX , &c less harmonious than AC , CE , &c, for the very same reasons as before.

Thirdly, any imperfect unisons will be changed into imperfect v^{th} s, whose vibrations are Ac and AD , (or AC and Ad) that is $2ab$ and $3AB$, by intermitting $2-1$ pulses of the series a , b , c , d , &c, which depresses the acuter unison an $viii^{th}$ lower, and $3-1$ pulses of the series A , B , C , D , &c leaving single ones between, which depresses the graver unison a xii^{th} or $viii + v^{th}$ lower; and thus the interval of the new sounds is an imperfect v^{th} , as represented in the uppermost parallel in the figure.

Now in the period where those intermissions leave the pulses r , t , w , y , &c, R , U , Y , &c, (as in the 4^{th} parallel) the intermediate ones will be taken away, and then Rr being the lesser of the two dislocations in the short cycle $R\gamma$ which includes z , is the least of all in this period. And the short cycles $R\gamma$, &c, on this side of Rr , will be more harmonious than AG , &c (in the first parallel); and on the other side, the short cycles RL , &c, will be less harmonious than AG , &c: For the same reasons as above.

Likewise in the period where the pulses q , s , u , x , &c, Q , T , X , &c are left (in the 5^{th} parallel), the intermediate ones will be absent, and then Qq is the least dislocation in this period, and a greater difference than before will be found in the harmony of the short cycles on each side of

R

Qq

Qq and Aa ; the difference $Xx - Qq$ being less than $Yy - Rr$ in the former case.

Lastly in the period where p, r, t, w , &c, P, S, W , &c, are left (in the lowest parallel), the intermediate ones are intermitted, and then Pp is the least dislocation in this period, and a difference still greater will be found in the harmony of the short cycles on each side of Pp and Aa , for the like reason. And the greatest difference will be found where these dislocations are equal when possible; that is, when a periodical point z bisects a short cycle of any consonance, which consists of any odd number of those of the unisons; and also when either of the coincident pulses at the ends of the complex or simple cycles of the unisons, bisects a short cycle of any consonance consisting of any even number of those of the unisons as in Fig. 35. Plate XII. The like proof is plainly applicable to the vibrations AC, Ad , or to those of any other consonance. Q. E. D.

Coroll. Hence any two imperfect consonances will be as equally harmonious as they possibly can be, when the periods (between their successive beats) which are bisected by their coincident pulses, are made equally harmonious; these periods having a mean degree of harmony among those of all the other periods in each consonance.

All those degrees of harmony occur in practical music, and whether sensibly different or
not,

not (o), must be used as if they were equal, and in theory we must take the medium among them.

As the proof of this conclusion has been pretty long, I avoided it in the Book by a paragraph in the demonstration of prop. XIII, which may now be proved somewhat differently.

*Another demonstration of Prop. xiii.
and its third corollary.*

Pl. xxv. Fig. 60, 61. Let op and OP represent the times of the single vibrations of imperfect unisons; ab and AB those of other imperfect unisons; o and O , a and A their coincident pulses; and if $ab = op$, the period of the pulses of the former unisons, will be to that of the latter, ultimately as bB to pP (p).

1. Taking bB to pP as 1 to 2, this is now the ratio of the lengths of the periods of the unisons op and OP , ab and AB ; and the latter is of the same length as the period of the least imperfections of octaves, whose single vibrations are ab and AC or $2AB$, by intermitting 2—1 pulses of the series A, B, C, D , &c, by prop. VIII.

Now the shorter length of the short cycles of the unisons op, OP , is $op = ab$, and that of the short cycles of the imperfect octaves is ac or $2ab$,

R 2

and

(o) Prop. xi. schol. 4. art. 5. last paragr.

(p) Cor. 8. lem. to prop. ix, and prop. xi. schol. 1.

and the ratio of their lengths is 1 to 2, which being the same as that of the periods of the unisons and octaves, shews that their short cycles are equally numerous in them.

The longer length of the short cycle of the octaves is AC or $2AB$, and the difference of the lengths is $2AB - 2ab = 2bB = cC$ the dislocation of the pulses at the end of the first short cycle, and is equal to pP , because we took $bB : pP :: 1 : 2$; therefore the several dislocations eE , &c, qQ , &c, at the ends of the subsequent short cycles of the octaves and unisons, are equal respectively throughout their half periods, which are therefore equally harmonious :

Because those dislocations are the causes that spoil the harmony, more or less according as they are greater or smaller; and causes constantly equal must have equal effects: And because the harmony of these half periods is the medium among the degrees of harmony of all the rest, by the coroll. to the lemma.

2. Fig. 60, 62. Again, taking bB to pP as 1 to 3, this is now the ratio of the periods of the imperfect unisons op and OP , ab and AB ; and the latter period is equal to that of imperfect xii^{th} s, whose vibrations will be ab and AD or $3AB$ by intermitting $3-1$ pulses of the series A, B, C, D , &c, so as to leave single pulses between every intermission (q). And since $Pp = (3bB =) dD$, it appears that the several subsequent dislocations qQ , &c, gG , &c, of the unisons

(q) Prop. VIII.

fons and xii^{th} are equally numerous and equal respectively throughout the half periods on each side of oO and aA ; which render the consonances as equally harmonious as they possibly can be, for the reasons above mentioned.

3. Fig. 60, 63. Lastly, taking bB to pP as 1 to 2×3 , this is now the ratio of the lengths of the periods of the dislocations of the imperfect unisons op and OP , ab and AB , for the reason above. And the latter period is of the same length as that of imperfect v^{th} , whose single vibrations $2ab$ and $3AB$ result from intermitting 2—1 pulses of the series a, b, c, d, e, f, g , &c, and 3—1 pulses of the series A, B, C, D, E, F, G , &c, so as to leave single pulses at the beginning, and between every intermission, by prop. VIII.

Now the shorter length of the short cycle of the unisons op , OP is $op = ab$, and that of the short cycle of the imperfect v^{th} is $2 \times 3ab$ (because $2ab : 3ab :: 2 : 3$) and the ratio of these lengths is 1 to 2×3 , the same as that of the periods of the imperfect unisons op , OP and the v^{th} , whose short cycles op and ag are therefore equally numerous in them.

The longer length of the imperfect short cycle of the v^{th} is $2 \times 3AB$ (because $2AB : 3AB :: 2 : 3$) and the difference of the longer and shorter lengths is $2 \times 3AB - 2 \times 3ab = 2 \times 3 \times \overline{AB - ab} = 2 \times 3bB = gG$, the dislocation of the pulses at the end of that short cycle, and is equal to pP ,

R 3

because

because we took $bB : pP :: 1 : 2 \times 3$. Therefore the several dislocations nN , &c, qQ , &c at the ends of all the subsequent short cycles of the v^{th} and unisons, are respectively equal in magnitude and number too, throughout the half periods on each side of the coincident pulses aA , oO ; which equalities make these consonances as equally harmonious as they possibly can be, for the reasons above.

4. Instead of the terms 2 and 3 of the ratio of the vibrations of perfect v^{th} , if we substitute those of any other perfect consonance, or m and n indeterminately for them, the method of demonstration will be evidently the same as in the last example.

Now those imperfect consonances of $viii^{\text{th}}$, xii^{th} , v^{th} , &c are not only equally harmonious with the same imperfect unisons op , OP , but also with one another; the dislocations pP , cC , dD , gG , at the ends of their first and subsequent short cycles, being equal and equally numerous in their periods. And since any one of them is equally harmonious to another of the same name at any other pitch, when their short cycles are equally numerous in their periods (r), it appears that all sorts of imperfect consonances are as equally harmonious as possible, when their short cycles are equally numerous in the periods of their imperfections. Q. E. D.

The equal harmony of flat consonances is demonstrable in the same manner.

Coroll.

(r) Prop. XII.

Coroll. Hence when imperfect consonances are equally harmonious, their temperaments have very nearly the inverse ratio of the products of the terms expressing the ratios of the single vibrations of the perfect consonances.

This is the third corollary to prop. XIII and may be demonstrated in this other manner.

The interval of the sounds of imperfect unisons is the temperament of the interval of any consonance whose single vibrations are different multiples of the vibrations of those unisons (*s*).

Now in all the examples of tempered consonances we made the vibration *ab* constant and *AB* variable. Consequently the several intervals of these imperfect unisons, or the logarithms of the ratios of *ab* to *AB* were very nearly proportional to the differences *bB* (*t*), which in the VIIIths and Vths were made equal to $\frac{1}{1 \times 2} pP$ and $\frac{1}{2 \times 3} pP$ respectively. Therefore when these consonances are equally harmonious, the ratio of their temperaments is $\frac{1}{1 \times 2}$ to $\frac{1}{2 \times 3}$ very nearly.

And when either of them is equally harmonious to another of the same name at a different pitch, their temperaments are equal (*u*), and the terms of the ratio of the vibrations of the

R 4

(*s*) Prop. VIII. cor. I.

(*t*) Cor. I. lem. to prop. IX.

(*u*) Prop. XII. coroll.

the perfect consonances of that name are the same.

Consequently the direct ratio of the temperaments and the inverse ratio of the products of those terms, are very nearly the same in all equally harmonious consonances.

An illustration of coroll. 1. Prop. xiii.

Pl. xxiv. Fig. 57. Let the intervals of the equidistant points $A, I, II, III, \&c$ be the longer or the shorter lengths of the imperfect short cycles of any given consonance; whose half period is AP ; A and a its coincident pulses; ab the lesser of the vibrations of the imperfect unisons whose half period is also AP . Make the perpendicular $PQ = \frac{1}{2}ab$, and draw AQ cutting the perpendiculars at $I, II, III, \&c$, in $D, D, D, \&c$. Then are these perpendiculars equal to the dislocations of the pulses between the successive short cycles of the imperfect consonance, by prop. vii and viii.

Fig. 58. Make the like construction denoted by the greek letters for any other imperfect consonance of the same or a different name. And if it be equally harmonious to the former, its half period $\alpha\pi$ will contain the same number of short cycles as AP does (x); suppose 6 in each. By lessening its temperament, let its half period be lengthened to αp , where erecting the perpendicular

(*) Prop. xii, xiii.

dicular $pq = \pi\kappa$ join αq cutting all the intermediate perpendiculars in $e, e, \&c.$ Then the several new dislocations $1e, 2e, 3e, \&c.$ will be smaller than $1\delta, 2\delta, 3\delta, \&c.$ respectively. Therefore the short cycles $\alpha 6e$, contained in a part of the new half period αp , are not only more harmonious than the short cycles $\alpha 6\delta$, contained in the old half period $\alpha\pi\kappa$, or than $AVID$, but those in the remaining part $e678e$ continue the harmony in the new half period αp , after that of the old half period is quite extinguished by the beat at the end of it.

Coroll. Since only the corresponding short cycles of imperfect consonances can admit of a just comparison, one by one, in the order of their succession, beginning from the coincident pulses, or from their least dislocations next to the periodical points, (as explained in the demonstration of the lemma,) if the periods of two consonances contain unequal numbers of their short cycles, the comparison will be imperfect; which is another argument *à priori* for the truth of the XIIth and XIIIth propositions.

Scholium 1.

In any pure consonance (y) the short cycle contains but one vibration of the base, as in Fig. 61, 62, and the equal times between the pulses of the treble are never subdivided by any pulses of the base, except at the ends of the short

(y) Sect. III. art. 8.

short cycles; and here the dislocations cC , dD are considered and adjusted with the analogous ones in other pure consonances, by the XIIIth proposition.

But in any other consonance whose short cycle contains several vibrations of the base, the equal times between the pulses of the treble are subdivided by the pulses of the base, not only at the ends of the short cycles, but between them, as at D , K , &c, Fig. 63; where the consideration of the inequalities of the intervals cD and De , iK and Kl , &c, seems to have been neglected in the said proposition, but in reality is implied in it.

Fig. 63. For supposing the alternate pulses D , K , &c to be intermitted or taken away, those v^{ths} will be changed into XII^{ths} an octave lower than the XII^{ths} in Fig. 62; but will not be equally harmonious with them, as the v^{ths} were supposed to be before that intermission, till the dislocations, gG , nN , &c, in Fig. 63 be doubled; that the temperaments of both the XII^{ths} may be equal and their periods proportional to their vibrations and short cycles (z).

While the dislocations gG , nN , &c, remain doubled, restore the pulses D , K , &c, to their places, and now the intermediate inequalities $Dc—De$, $Ki—Kl$, &c are also the doubles of their former magnitudes and the new v^{ths} are less harmonious than the XII^{ths} in Fig. 62, and will not be equally harmonious with them till the
dislocation,

(*) Prop. XII. coroll.

dislocation, gG , nN , &c, and consequently the inequalities $Dc - De$, $Ki - Kl$, &c be contracted to their former magnitudes.

Therefore these interrupted consonances are not considered as pure ones in the XIIth and XIIIth propositions, but allowance is made on course for the effect of the intermediate pulses of the base.

Scholium 2.

Pl. xxv. Fig. 63. Supposing the letters d , k to be restored to the places of the absent pulses of the imperfect unisons, that fall next to D and K , I call the lines or times dD , kK the Aberrations of the interior pulses D , K , from the places d , k which they have in the perfect short cycles. Likewise in the upper part of the same figure, if AE and ad be the single vibrations of an imperfect 4th, then Ee is an aberration of one of the interior pulses of the base in the first short cycle.

Now if the ratio of the times of the single vibrations of any perfect consonance be m to n in the least integers, and when it is tempered, if $2D$ be the sum of the exterior dislocations in any given short cycle, the aberration or sum of the aberrations of the interior pulses of the base, from the places they have when the consonance is perfect, will be $\overline{n-1} \times D$.

The reason of the theorem will soon appear by drawing a short cycle or two of a 4th, III^d, &c,

&c, and by observing, that as n is the number of the vibrations of the base contained in any short cycle, so $n-1$ is the number of its pulses exclusive of the extremes, and that the sum of the exterior dislocations is equal to the sum of any two interior aberrations equidistant from them, or to double the aberration in the middle; as is plain from the arithmetical progression of the alternate lesser intervals of the imperfect unisons, from which the given consonance is derived.

Therefore in two equally harmonious consonances, as the sum of the exterior dislocations in any short cycle of the one, is to the sum of them in the corresponding short cycle of the other, in a certain constant ratio (a), so the interior aberration or the sum of the interior aberrations in the former short cycle, is to the sum of them in the latter in another constant ratio; and *componendo*, the totals of the exterior dislocations and interior aberrations are also in another constant ratio.

But the temperaments and periods of the two consonances must be adjusted by the first given ratio alone, without any regard to the second or third.

1. Because the exterior dislocations are of a different kind from the interior aberrations. For as in seeing so in hearing, it is more difficult for the sense to perceive the quantity of a small inequality in the larger successive interval of the points

or

(a) By the foregoing illustration.

or pulses *c*, *D*, *e*, Fig. 63, than to perceive the same or a different small quantity when bounded by two visible points or audible pulses *g*, *G*. And the difficulty is greater in more complex short cycles of imperfect 4^{ths}, 111^{ds}, 61^{ths}, &c, where the successive intervals between the points analogous to *c*, *D*, *e*, do not err from the simplest ratio of 1 to 1, but from the more complex ones of 1 to 2, 1 to 3, &c ; as will easily appear from the disposition of the pulses in such cycles in Fig. 5, Plate 1. For which reason the ratio of the sum of the interior aberrations ought not to be compared and compounded with that of the exterior dislocations.

2. Because it appears from the corollary to the foregoing Illustration, that no other regard can be had to the interior aberrations, than what follows on course from the given ratio of the exterior dislocations, determined by the equality of their numbers in the periods of the two consonances, as in prop. XII and XIII.

Scholium 3.

To give the reader more determinate ideas of the numbers of vibrations, short cycles and dislocations contained in the long cycles and periods of imperfect consonances, and of their absolute duration in practical music, I will add a computation of them in a consonance of 7^{ths} tempered

pered by $\frac{1}{4}$ comma, as it usually is, more or less, in organs and harpsichords.

Pl. XII. Fig. 34. If $AB : ab :: 322 : 321$, the interval of the sounds of these vibrations is $\frac{1}{4}$ comma nearly (*b*). Whence $321 AB = 322 ab = AZ$, is the length of the simple cycle of the dislocations of the pulses of the vibrations AB , ab , or of the period of the imperfections of any consonances whose vibrations are different multiples of AB and ab (*c*) and whose temperament is the interval of the sounds of AB and ab (*d*).

Now the vibrations of imperfect v^{ths} are AD and ac , or $3 AB$ and $2 ab$, and the two constant lengths of their imperfect short cycles are $AG = 2 AD = 2 \times 3 AB$ and $ag = 3 ac = 3 \times 2 ab$.

$$\text{Hence } AZ = 321 AB = \frac{321}{3} \times 3 AB = 107 AD = \frac{321}{6} \times 6 AB = 53 \frac{3}{6} AG;$$

$$\text{Likewise } AZ = 322 ab = \frac{322}{2} \times 2 ab = 161 ac = \frac{322}{6} \times 6 ab = 53 \frac{4}{6} ag.$$

And after the coincidence of the pulses, their first dislocation is $gG = \frac{1}{161} AD$; and the limit of the greatest dislocation is $\frac{1}{2} ab = \frac{107}{644} AD$.

For

(*b*) Prop. XI. schol. 4. art. 6.

(*c*) Prop. VIII.

(*d*) Prop. VIII. cor. I.

For $AB : AB - Ab$, or $bB :: 322 : 1$,
 whence $Bb = \frac{1}{322} AB$, and $gG = 6bB = \frac{6AB}{322}$
 $= \frac{3AB}{161} = \frac{1}{161} AD$, and $\frac{1}{2} ab = \frac{1}{2} \times \frac{321}{322} AB =$
 $\frac{1}{2} \times \frac{321}{322} \times \frac{1}{3} AD = \frac{107}{644} AD$, and is the limit of
 the greatest dislocation, or alternate lesser inter-
 val of the pulses of AB , ab in any half period,
 by prop. VII.

Now by an experiment mentioned in prop.
 XVIII, I found that the particles of air in an or-
 gan pipe called *d* or *d-la-fol-re*, in the middle of
 the scale of the open diapason, made 262 com-
 plete vibrations or returns to the places they went
 from, and consequently propagated 262 pulses of
 air to the ear (*e*) in one second of time; though
 the pitch of the organ was above half a tone
 lower than the present pitch at the Opera. And
 taking that sound for the base of our v^{th} , whose
 vibration AD represents a certain quantity of
 time, we have $262 AD = 1$ second and hence
 the absolute times AZ , AG , Gg and $\frac{1}{2} ab$ are the
 following fractions of $1''$.

For, $262 AD : AZ$ or $107 AD :: 1'' : \frac{107}{262} \times 1''$.

and $262 AD : AG$ or $2 AD :: 1'' : \frac{1}{131} \times 1''$.

and $262 AD : Gg$ or $\frac{1}{161} AD :: 1'' : \frac{1}{262 \times 161}$

$\times 1'' = \frac{1}{42182} \times 1''$.

and

(*e*) Sect. I. art. 12.

$$\text{and } 262 AD : \frac{1}{2} ab \text{ or } \frac{107}{644} AD :: 1'' : \frac{107}{262 \times 644} \\ \times 1'' = \frac{1}{1568} \times 1''.$$

And the reciprocal of the periodical time AZ , = $\frac{107}{262} \times 1''$, is the number of periods and also of beats in $1''$ (f) namely $\frac{262}{107} = 2.45$ nearly in $1''$, or 245 in $100''$ nearly.

And the least dislocations in the short cycles, as $\Delta \varepsilon \lambda K$, which include the successive periodical points Z , are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ of the dislocation Gg next to the coincident pulses.

And these measures are to those in any other v^{th} s in the scale of that organ, in the given ratio of the times of the single vibrations of their bases. And the like measures in any other given consonance, whose temperament is given, may be computed in the like manner, or derived from these by the corollaries to prop. ix.

In this example the v^{th} s were tempered sharp, and when they are equally tempered flat, by taking Ad and AC for the single vibrations, the computation and the measures will be but very little different.

(f) Prop. x.

Scholium

Scholium 4 to prop. xx.

Tables and observations on the numbers of beats of the concords in the principal systems.

The following table shews the number of beats made in 15 seconds, by the several concords to the base note D, or D-sol-re, at the Roman Pitch, and likewise the proportions of the beats of the same concords to any other base note; with this design, that persons wanting leisure or proper qualifications for examining the principles and conclusions requisite to determine the system of equal harmony, may yet form some judgment of its advantages and disadvantages, when compared with the system of mean tones and that of Mr. *Huygens*; upon this allowed principle, that, *cæteris paribus*, concords to the same base are more or less disagreeable in their kind for beating faster or slower respectively.

To assist the reader's judgment I have added the following observations resulting from inspection of the first table.

1. The beats in column 1 differ less from one another than those in column 2 and 3 do, agreeably to the Name of the system.

2. The beats of the v , $v + viii$, $v + 2viii$ in col. 1, are a little quicker than those in col. 2, in the ratio of 10 to 9, and quicker than those in col. 3, in a ratio something greater. But the

S

beats

beats of the VI, VI + VIII, VI + 2VIII in col. 1 are much slower than those in col. 2 and 3, in the ratio of 2 to 3 and more.

3. These quick beats of the VIth in col. 2 and 3 have the disadvantage to be doubled in every

TAB. I.

The beats in 15" of all the - - - - -

VI + 2VIII. $\frac{3}{20}$	80	120	133 $\frac{1}{11}$
V + 2VIII. $\frac{1}{6}$	40	36	34 $\frac{8}{11}$
III + 2VIII. $\frac{1}{5}$	13	0	4 $\frac{1}{4}$
VI + VIII. $\frac{3}{10}$	40	60	66 $\frac{6}{11}$
V + VIII. $\frac{1}{3}$	20	18	17 $\frac{4}{11}$
III + VIII. $\frac{2}{5}$	13	0	4 $\frac{1}{4}$
VI. $\frac{3}{5}$	20	30	33 $\frac{3}{11}$
V. $\frac{2}{3}$	20	18	17 $\frac{4}{11}$
III. $\frac{4}{5}$	13	0	4 $\frac{1}{4}$
System of	equal	mean	M. Huy-
	harm.	tones	gens.
Column	1	2	3

every superior octave, whereas the quick beats of the vth in col. 1. remain the same in the second octave and are only double in the third &c.

4. The beats of the III^d and compounds are quicker indeed in col. 1 than those in col. 2 and

S 2

3;

T A B. I.

----- concords to the base D-fol-re.

$3^d + 2VIII. \frac{5}{24}$	96	144	160
$4^{th} + 2VIII. \frac{3}{16}$	107	96	93
$6^{th} + 2VIII. \frac{5}{32}$	83	0	27
$3^d + VIII. \frac{5}{12}$	48	72	80
$4^{th} + VIII. \frac{3}{8}$	53	48	46
$6^{th} + VIII. \frac{5}{16}$	42	0	14
$3^d \cdot \frac{5}{6}$	24	36	40
$4^{th} \cdot \frac{3}{4}$	$26\frac{2}{3}$	24	23
$6^{th} \cdot \frac{5}{8}$	$20\frac{4}{5}$	0	7
System of	equal harm.	mean tones	M. Huy- gens.
Column	1	2	3

3; but being slower than the beats of the other concords in all the systems, they can scarce be so offensive as these will be.

5. Likewise the beats of the 6th and compounds in col. 1, being slower than those of the 4th

TAB. II.

The order of the harmony - - - - -

VI + 2VIII	240	360	$399\frac{3}{11}$
V + 2VIII	40	36	$34\frac{8}{11}$
III + 2VIII	13	0	$4\frac{1}{4}$
VI + VIII	120	180	$199\frac{7}{11}$
V + VIII	20	18	$17\frac{4}{11}$
III + VIII	26	0	$8\frac{1}{2}$
VI	60	90	$99\frac{9}{11}$
V	40	36	$34\frac{8}{11}$
III	52	0	17
System of	equal harm.	mean tones	M. Huy- gens.
Column	I	2	3

4th and 3^d and compounds with the same number of VIII^{ths} in all the systems, can hardly offend the ear so much as the quicker beats of these other concords will.

6. The sums of the beats of all the concords
to

T A B. II.

----- of all the concords.

3 ^d + 2VIII	480	720	800
4 th + 2VIII	321	288	279
6 th + 2VIII	415	0	135
3 ^d + VIII	240	360	400
4 th + VIII	159	144	138
6 th + VIII	210	0	70
3 ^d	120	180	200
4 th	80	72	69
6 th	104	0	35
System of	equal harm.	mean tones	M. Huy- gens.
Column	I	2	3

to the note D-sol-re in both the col. 1, 2, 3, are respectively 759, 702, 805, whose proportions with respect to the same or any other base note are 38, 35, 40 very nearly. The small excess of the first sum above the second arises chiefly from the beats of the III^d and 6th with their compounds, which in all probability are inoffensive, as we said before.

But a completer rule for comparing the harmony of imperfect concords to a given base, appears in the second table; *That concord to which a smaller number corresponds, being more harmonious, in its kind, than any other to which a larger number corresponds*; which affords two or three more observations.

7. In col. 1 and 3, the III^{ds} become more harmonious by the addition of VIII^{ths}.

8. In all the systems, the v + VIII is more harmonious than the v and v + 2VIII.

9. In col. 1, the vth is more harmonious than the III^d and VIth, the v + VIII than the III + VIII and VI + VIII, and likewise the 4th and compounds than the 6th and 3^d and compounds with equal numbers of VIII^{ths}.

10. It may be objected to the system of equal harmony that the beats of the v^{ths} are not only a little quicker, but something stronger and distincter than those of the other concords; which deserves to be considered. On the other hand it should be considered too, whether those very quick and less distinct beats of the VIth and compounds,

pounds, have not a worse effect in destroying the clearness of their harmony.

These are the principal advantages and disadvantages that occur in comparing these systems. For as to the false concords being something worse in the system of equal harmony than in the other two (*g*), this is no objection to the system, but only to the application of it to defective instruments; and I have shewn above how to supply their defects, without the least inconvenience to the performer (*b*).

I shall only observe, that the first table was calculated from the temperaments of the systems in prop. xvii and scholium, by the corollaries to prop. xi; and that the numbers in that table multiplied by the numerators of the known fractions, annexed to the characters of the corresponding concords, produce the corresponding numbers in the second table, according to coroll. 12, prop. xiii.

Scholium 5 to prop. xx.

The sound of an open metalline pipe will be flattened or sharpened a little by bending a small part of the metal at the open end, a little inwards or outwards, respectively.

The sound of a stopt pipe, made of metal, will be flattened or sharpened a little by bending the ears, at the sides of the mouth, a little inwards or outwards, respectively.

The

(*g*) Sect. viii. art. 2.

(*b*) Sect. viii. art. 11th.

The sound of an open wooden pipe will be flattened or sharpened a little, by depressing or raising the leaden plate that hangs over the open end, respectively.

The sound of a stopt wooden pipe will be flattened or sharpened by drawing the plug outwards or forcing it inwards, respectively.

The sound of a reed-pipe will be flattened or sharpened by causing the brazen tongue to be lengthened or shortened, respectively.

There is something curious in the reasons of these effects, but as they cannot be well explained in few words, I must omit them.



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Corrections.

- Page 6. note, lin. 10, for ὁξυτέρον, r. ὁξυτέρον.
 11. lin. 4, for *cé*, r. *cc*.
 51. note, lin. 2, *delete* the first *minus*—
 69, 71, 73 in the running titles, for
 Prop. VIII. r. Lemma.
 71. lin. penult. for $-\frac{q}{p}a$, r. $-\frac{q}{p}d$.
 114, 115, notes (*p*) and (*q*), for cor. 7. r.
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 168. lin. 1. *delete* 9.
 169. lin. 12. for single, r. one.
 163, 165, 167, 169, 171, in the running
 titles, for Prop. XVII. r. Art. 5, 7,
 9, 12, 15, respectively.
 227. note (*k*) for schol. r. schol. 4.

1870
The first of the year
was a very dry one
and the crops were
very poor.

The second of the year
was a very wet one
and the crops were
very good.

The third of the year
was a very dry one
and the crops were
very poor.

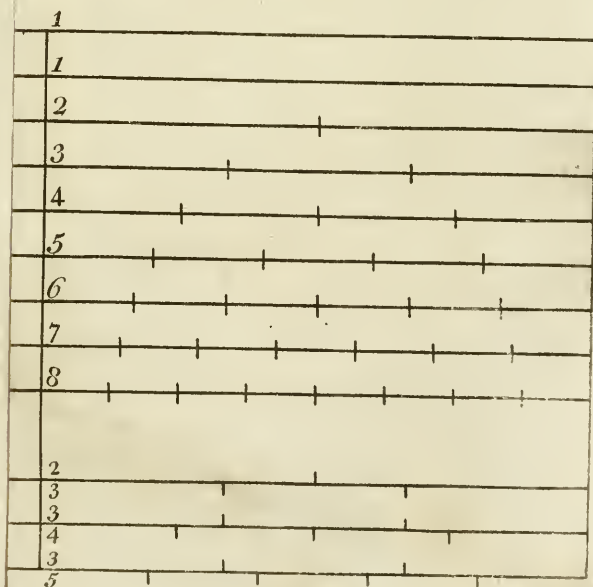
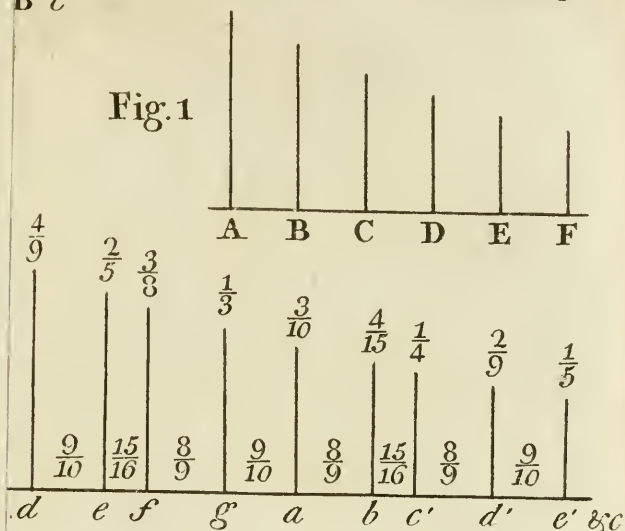
The fourth of the year
was a very wet one
and the crops were
very good.

The fifth of the year
was a very dry one
and the crops were
very poor.

Fig. 2

$\frac{8}{15}$ $\frac{1}{2}$
B c

Fig. 1



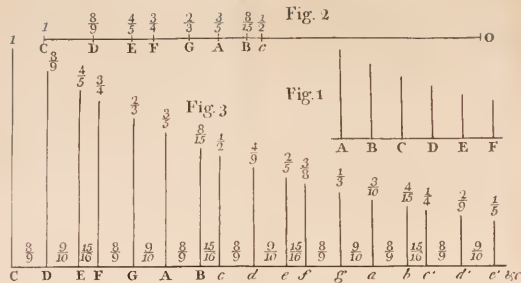
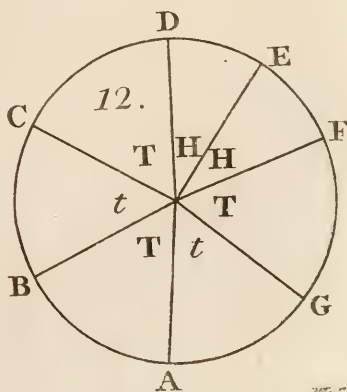
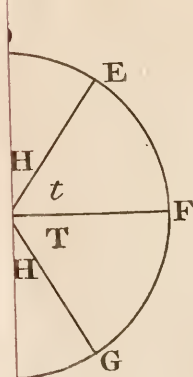
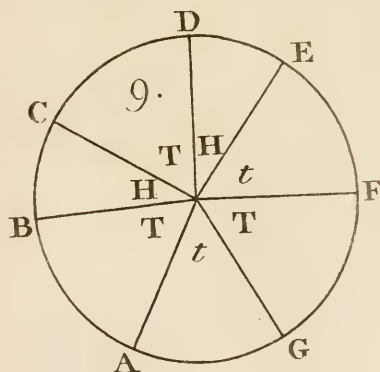
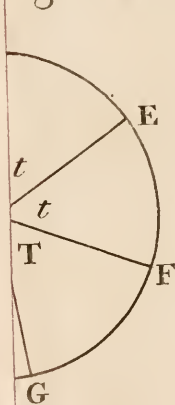
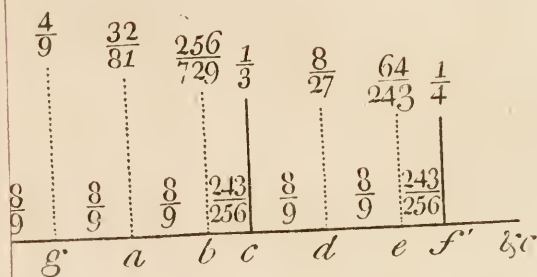


Fig. 6.



W.S. f.c.

Fig. 6.

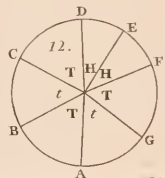
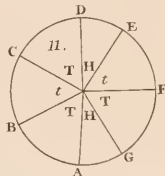
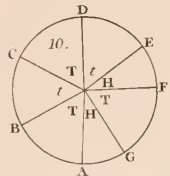
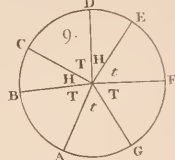
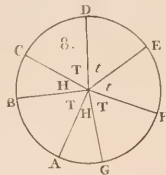
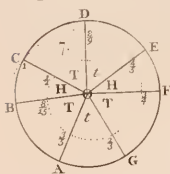
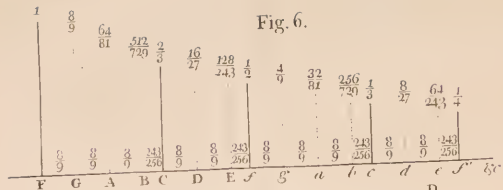


Fig. 14.

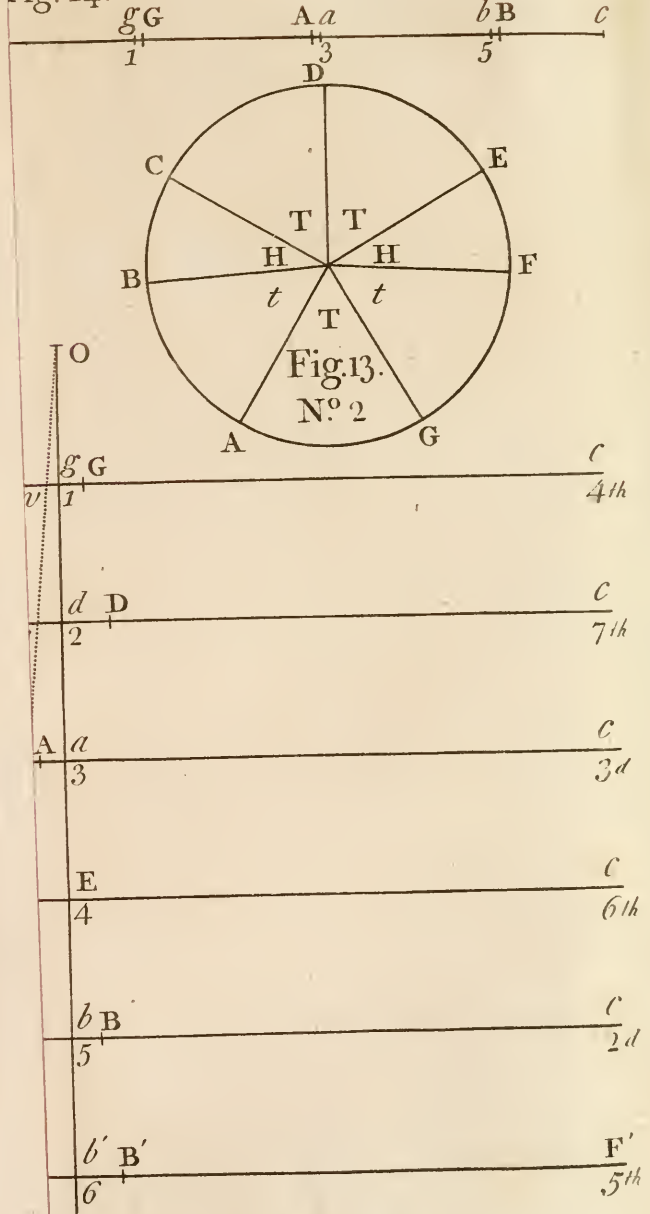


Fig. 14.

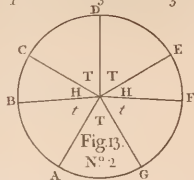
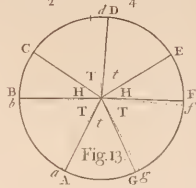
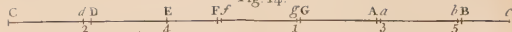
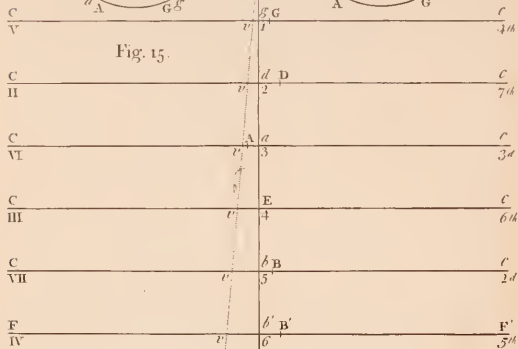


Fig. 15.



PL. IV.

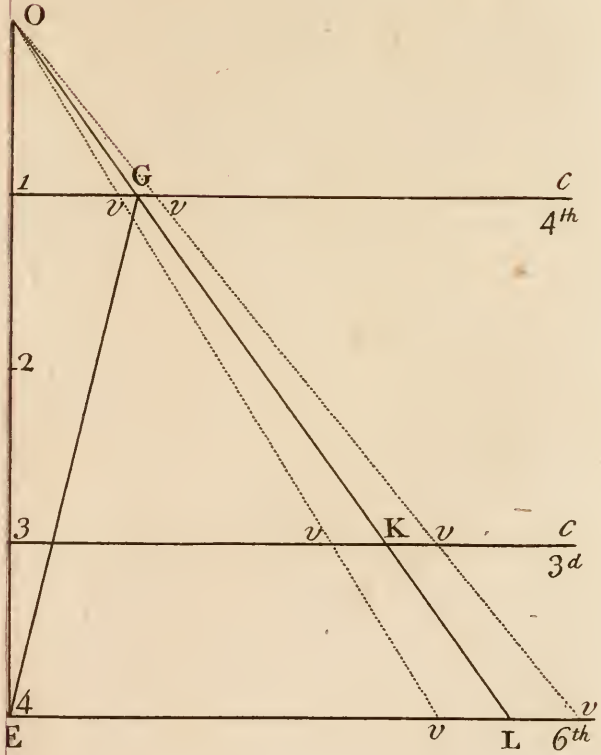
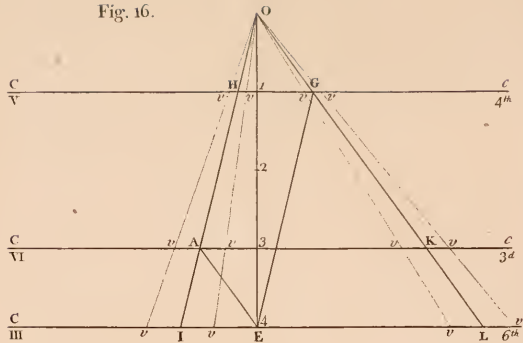
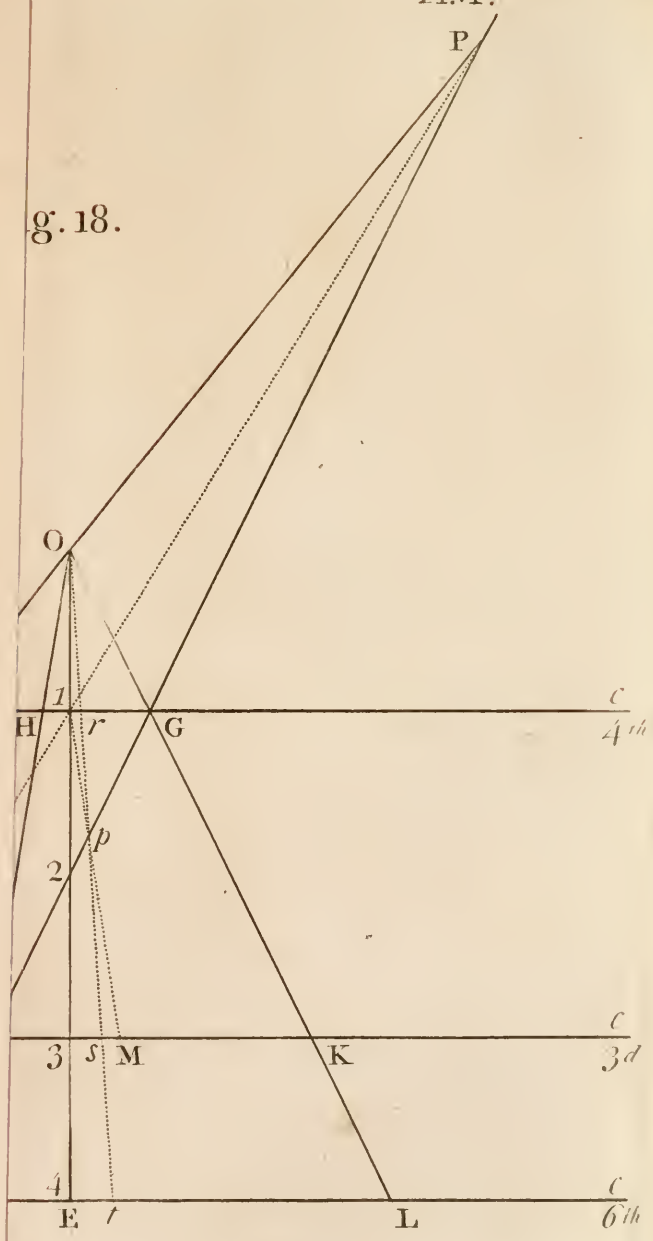


Fig. 16.



g. 18.

Pl.M.



W.S.

Fig. 18.

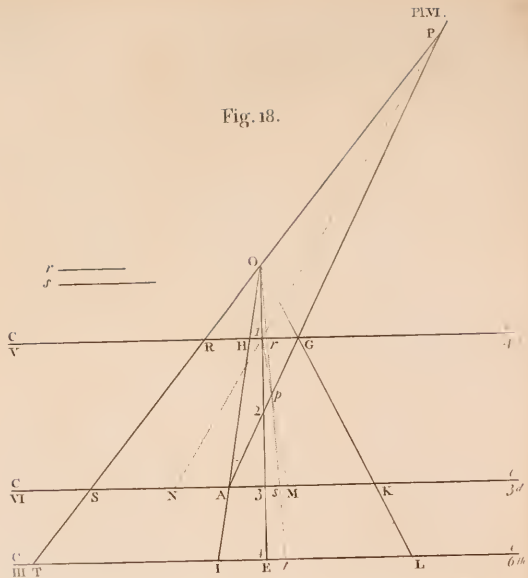
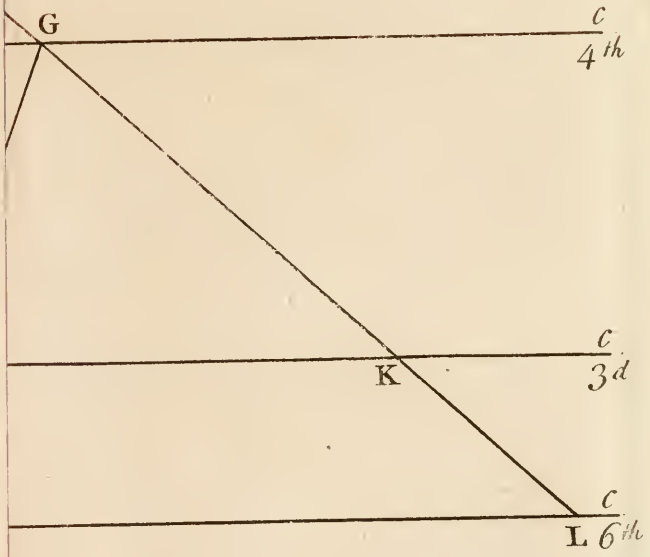


Fig. 19.



r _____

t _____

Fig. 19.

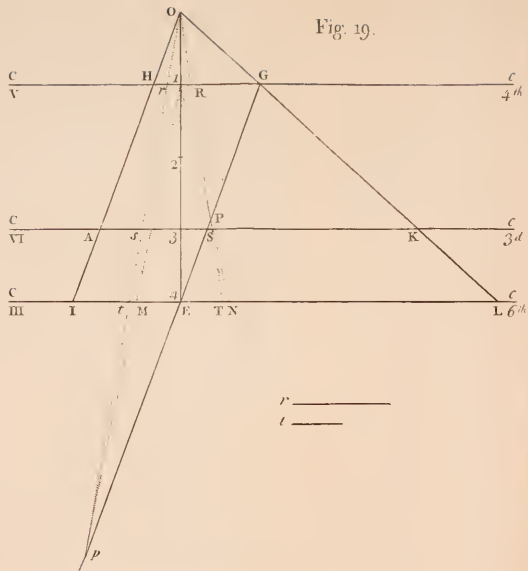


Fig. 20.

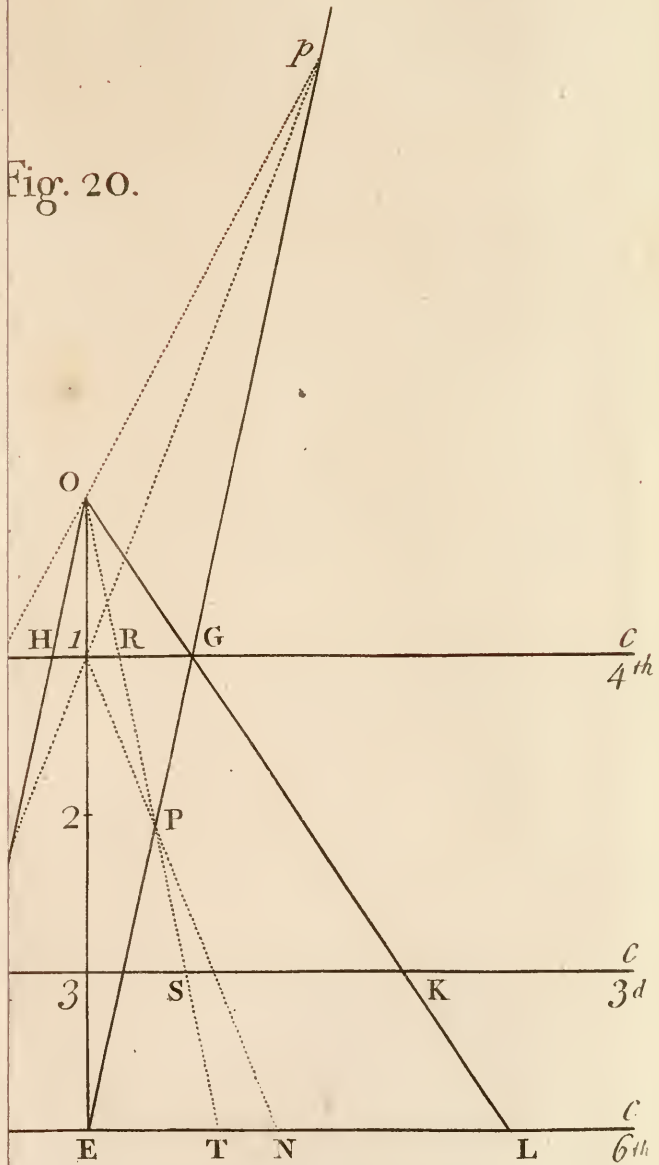


Fig. 21.

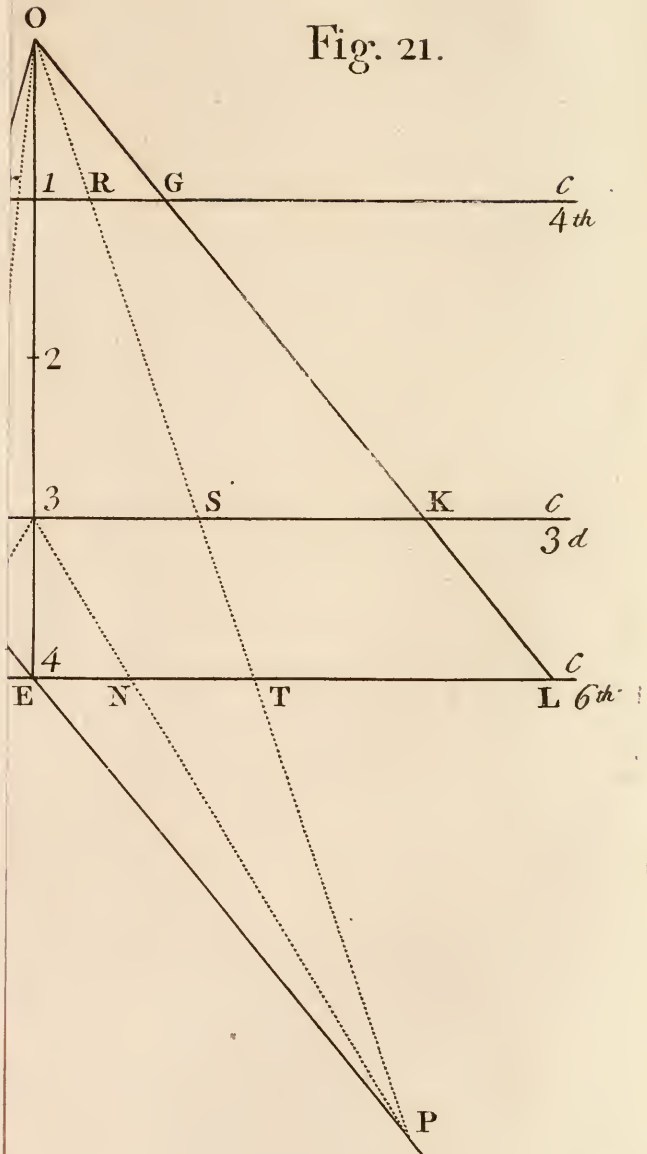


Fig. 21.

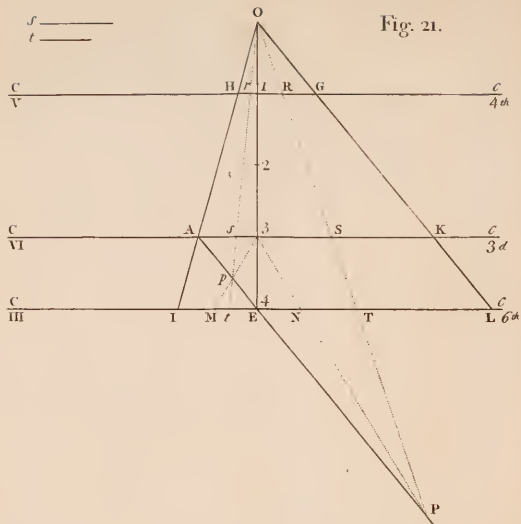


Fig. 22.

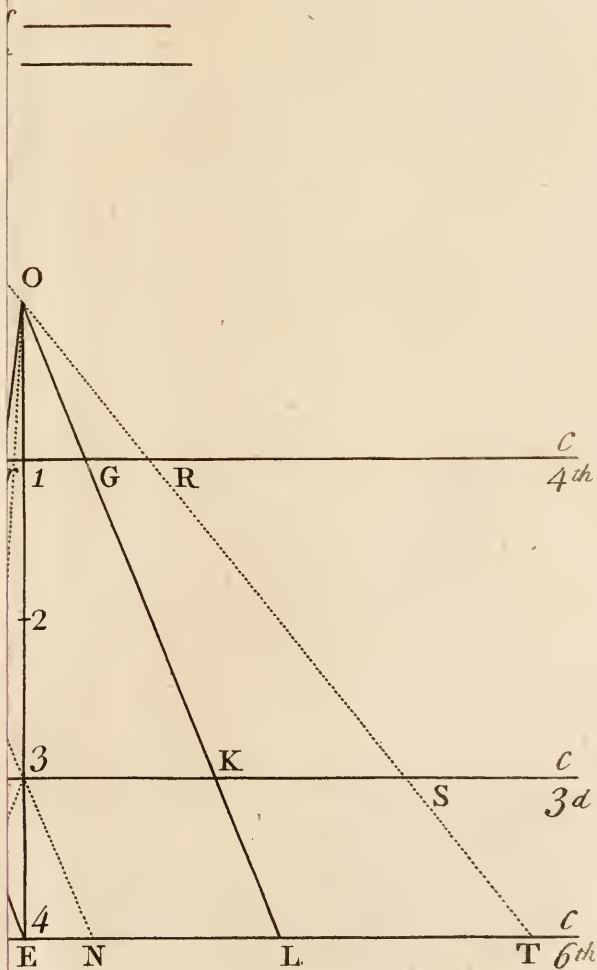
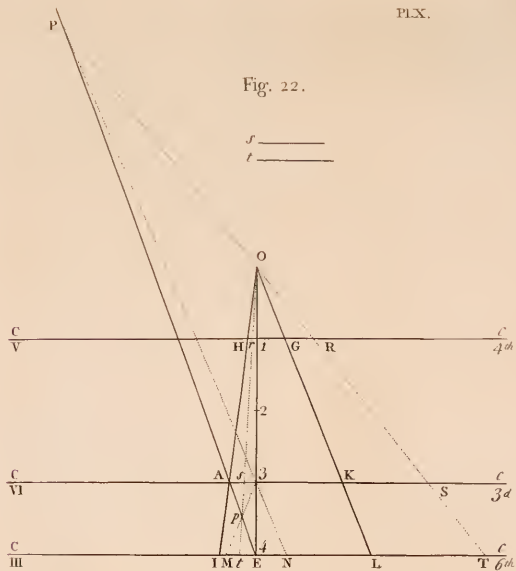


Fig. 22.

s ———
 t ———



PL. XI.

e *v*
E Q R S T U

e
E Q R S T U *x*

E Q R S T U *x* *y*

f
E Q R S T U W

e
E Q R S T U

e
E Q R S T U

W Stephens sc.

Fig.

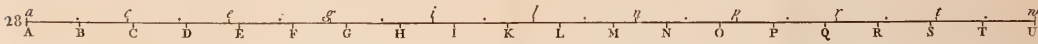
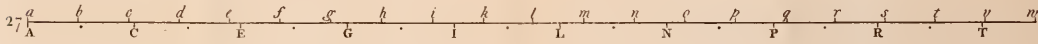
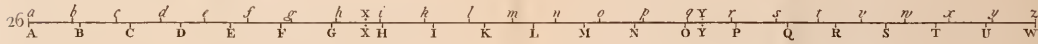
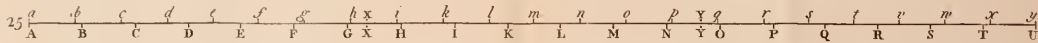
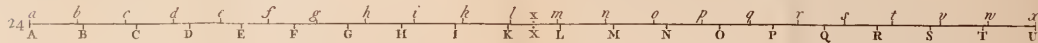
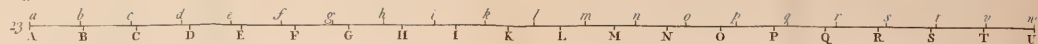
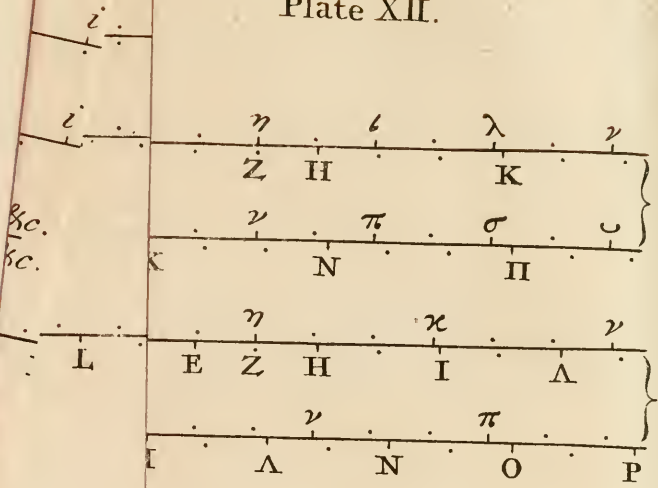


Plate XII.



α β

$n\nu$
 nV

$\frac{2}{1}$ $\frac{16}{9}$ $\frac{8}{5}$

C D E

VIII 7th 6th

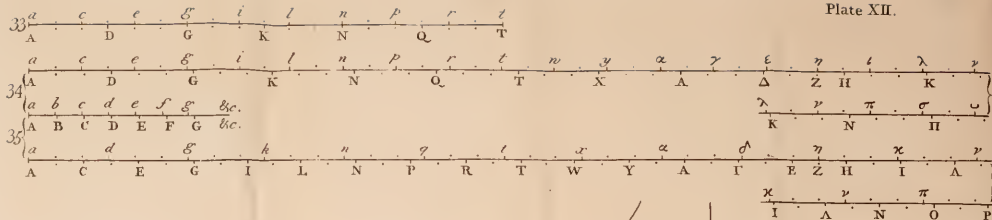
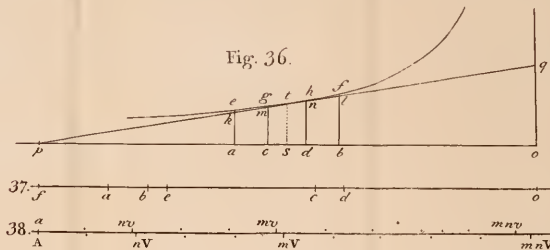


Fig. 36.



Tab. 1.

$\frac{2}{1}$	$\frac{16}{9}$	$\frac{8}{3}$	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{6}{5}$	$\frac{16}{15}$	1	$\frac{8}{9}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{8}{15}$	$\frac{1}{2}$	$h.c.$
C	D	E	F	G	A	B	c	d	e	f	g	a	b	c'	
VIII	7 th	6 th	V	4 th	3 ^d	2 ^d	γ	Π	III	4 th	V	VI	VII	8 th	

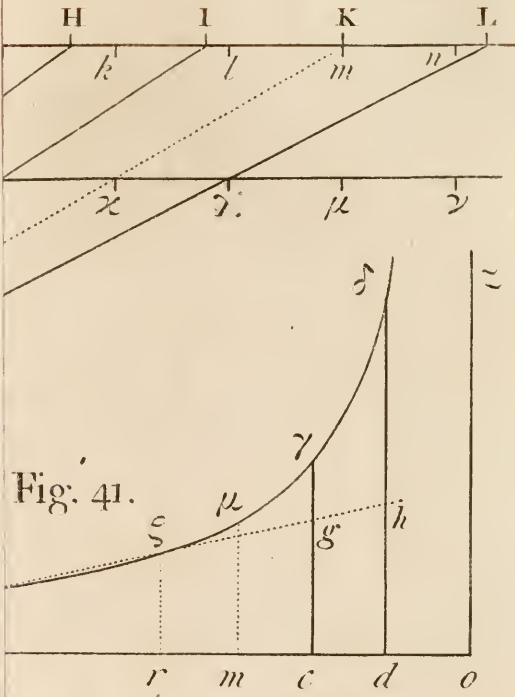


Fig. 41.

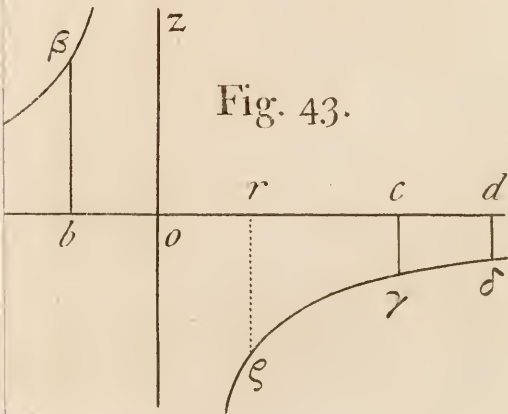


Fig. 43.

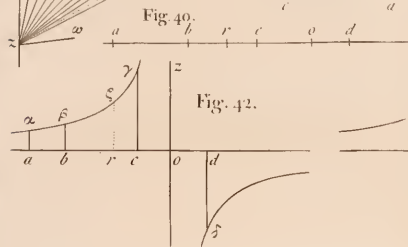
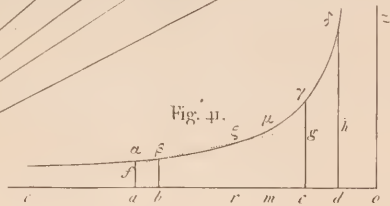
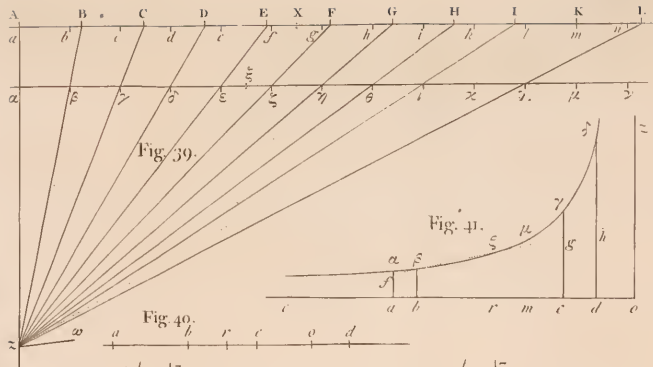


Fig. 42.

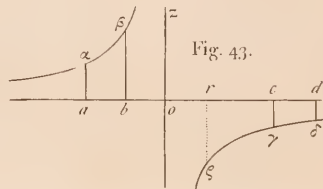


Fig. 43.

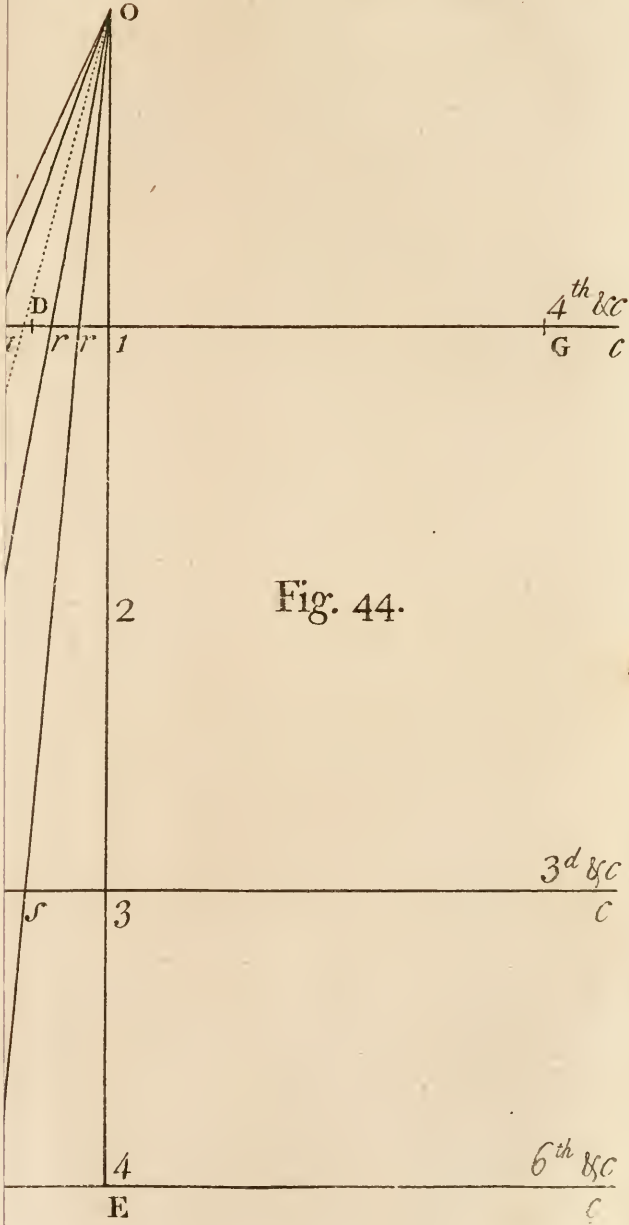


Fig. 44.

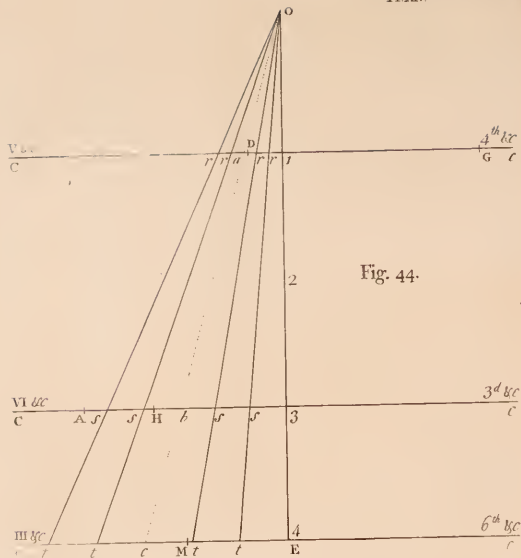
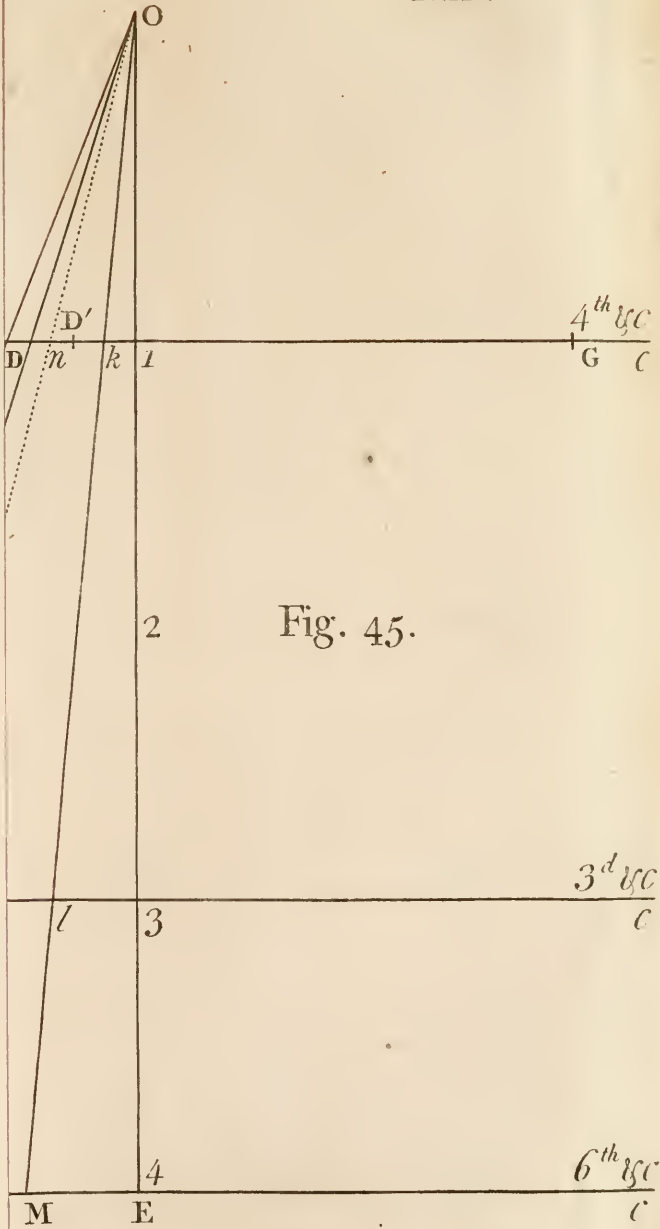


Fig. 44.



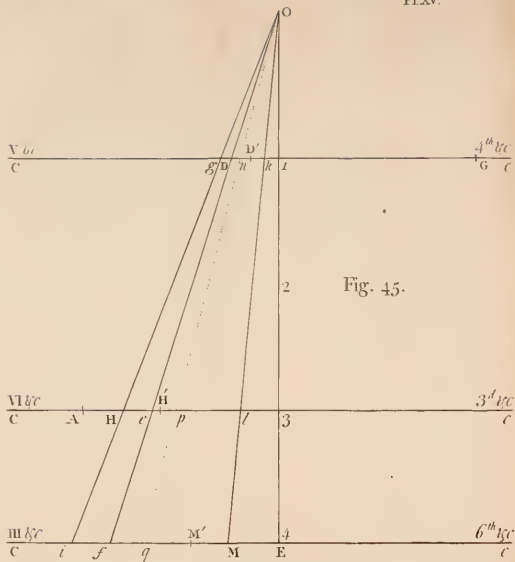


Fig. 45.

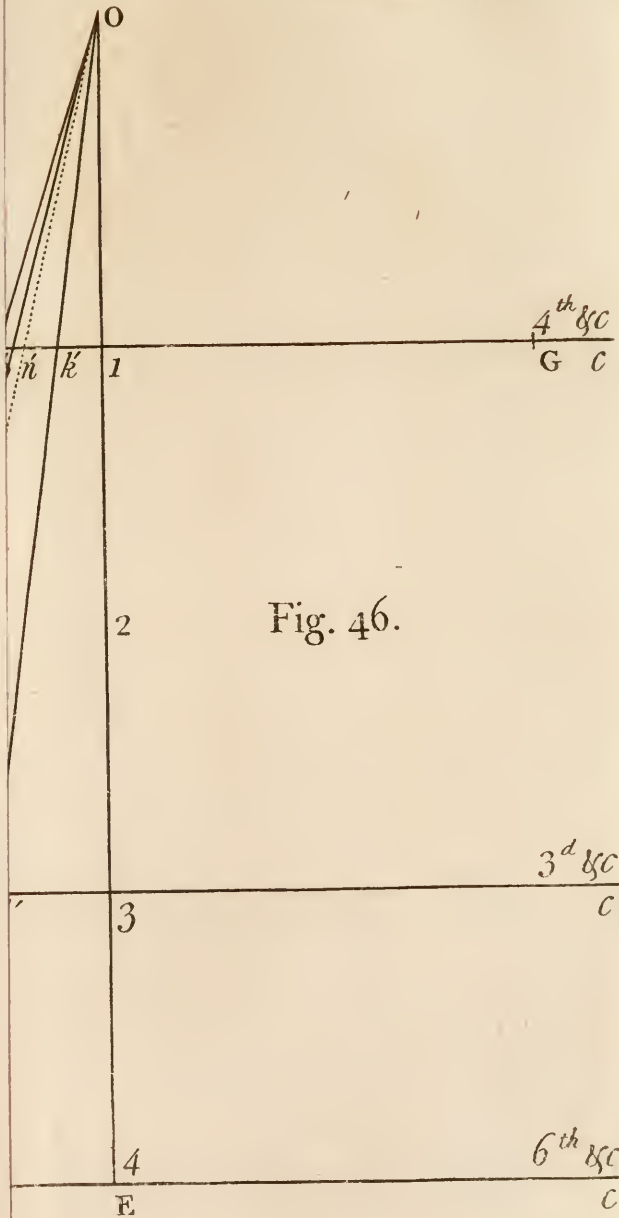
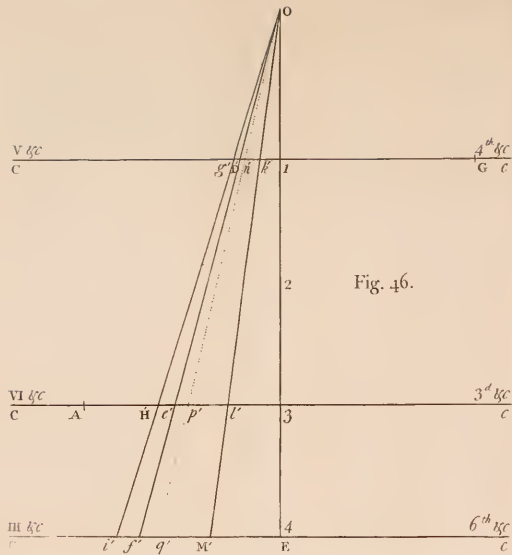
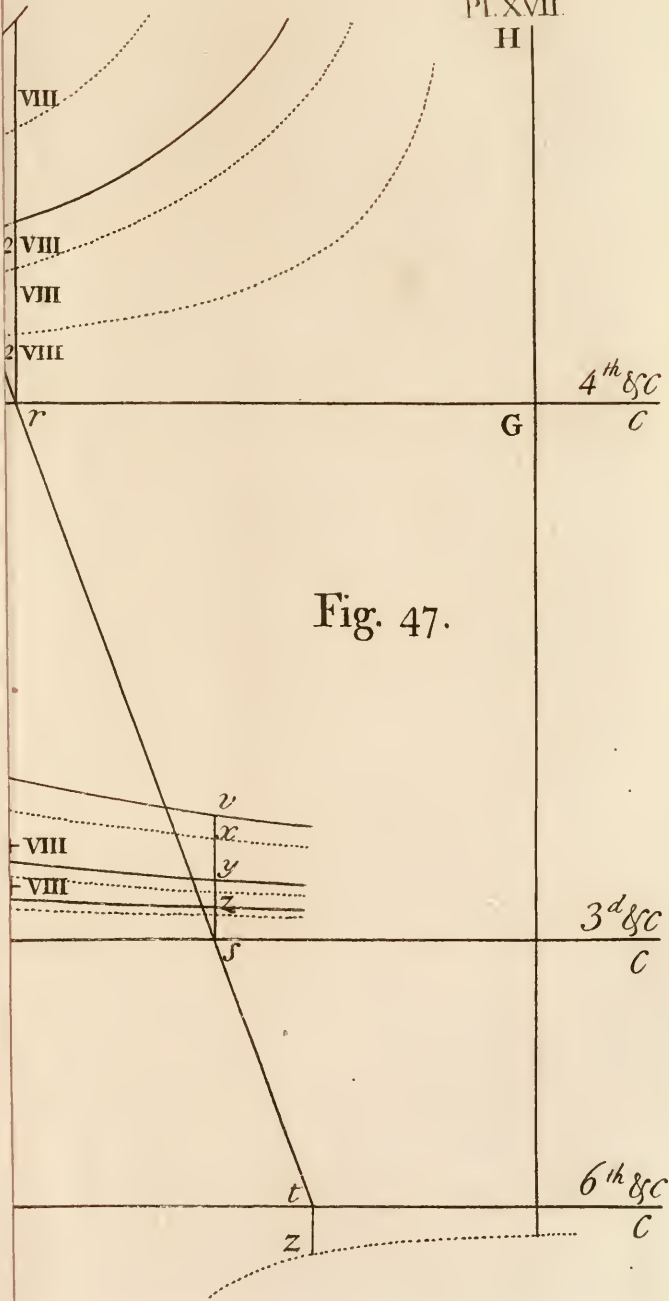
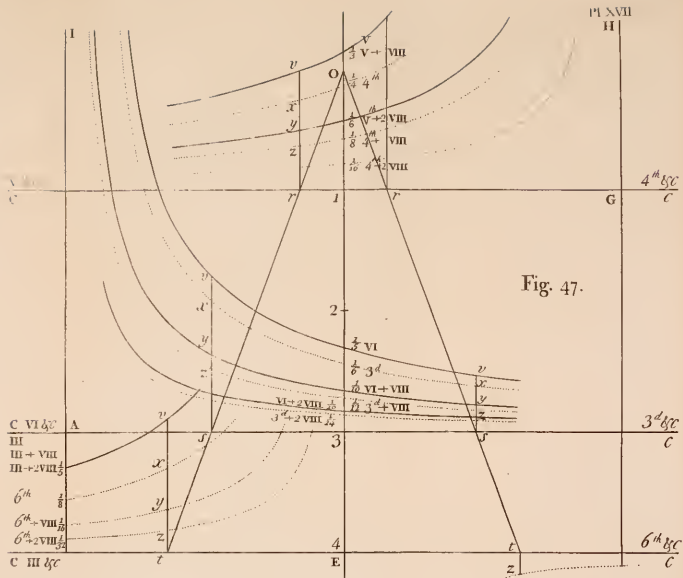


Fig. 46.







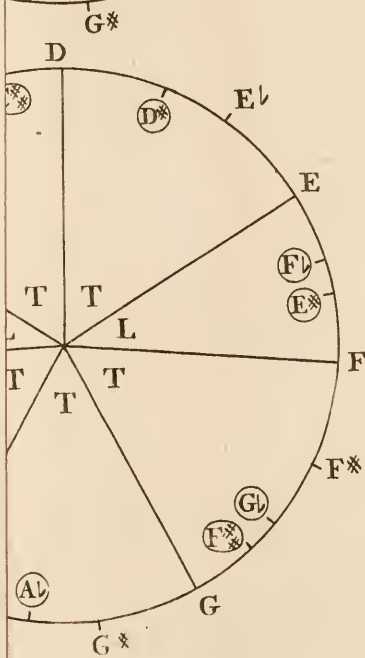
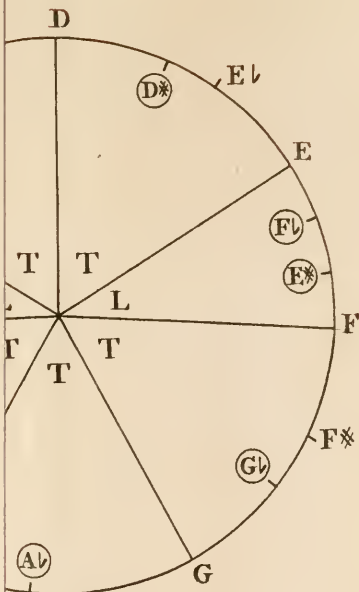
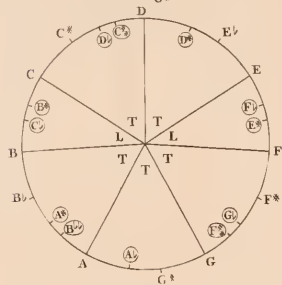


Fig. 48.



Fig. 49.



8 major Consonances to 24 Keys. Pl. XIX

4 th	IV	5 th	V	6 th	VI	7 th	VII	8 th
F*	.	G*	.	A*	.	B*	.	C*
B*	.	C*	C*	D*	.	E*	.	f*
E*	.	F*	F*	G*	.	A*	.	b*
A*	.	B	B*	C*	C*	D*	.	e*
D*	.	E	E*	F*	F*	G*	.	a*
G*	.	A	A*	B	B*	C*	C*	d*
C*	C*	D	D*	E	E*	F*	F*	g*
F*	F*	G	G*	A	A*	B	B*	c*
B	B*	C	C*	D	D*	E	E*	f*
E	E*	F	F*	G	G*	A	A*	b
A	A*	B [↓]	B	C	C*	D	D*	e
D	D*	E [↓]	E	F	F*	G	G*	a
G	G*	A [↓]	A	B [↓]	B	C	C*	d
C	C*	D [↓]	D	E [↓]	E	F	F*	g
F	F*	G [↓]	G	A [↓]	A	B [↓]	B	c
B [↓]	B	C [↓]	C	D [↓]	D	E [↓]	E	f
E [↓]	E	F [↓]	F	G [↓]	G	A [↓]	A	b [↓]
A [↓]	A	B [↓]	B [↓]	C [↓]	C	D [↓]	D	e [↓]
D [↓]	D	.	E [↓]	F [↓]	F	G [↓]	G	a [↓]
G [↓]	G	.	A [↓]	B [↓]	B [↓]	C [↓]	C	d [↓]
C [↓]	C	.	D [↓]	.	E [↓]	F [↓]	F	g [↓]
F [↓]	F	.	G [↓]	.	A [↓]	B [↓]	B [↓]	c [↓]
B [↓]	B [↓]	.	C [↓]	.	D [↓]	.	E [↓]	f [↓]
.	E [↓]	.	F [↓]	.	G [↓]	.	C [↓]	b [↓]
1	D	1	L	1	L	1	L	

Table of minor & major Consonances to 24 Keys. PI XIX

Key	2 ^d	II ^d	3 ^d	III	4 th	IV	5 th	V	6 th	VI	7 th	VII	8 th
C [♮]	D [♮]	.	E [♮]	.	F [♮]	.	G [♮]	.	A [♮]	.	B [♮]	.	C [♮]
F [♮]	G [♮]	.	A [♮]	.	B [♮]	.	C [♮]	C [♮]	D [♮]	.	E [♮]	.	F [♮]
B [♮]	C [♮]	C [♮]	D [♮]	.	E [♮]	.	F [♮]	F [♮]	G [♮]	.	A [♮]	.	B [♮]
E [♮]	F [♮]	F [♮]	G [♮]	.	A [♮]	.	B	B [♮]	C [♮]	C [♮]	D [♮]	.	E [♮]
A [♮]	B	B [♮]	C [♮]	C [♮]	D [♮]	.	E	E [♮]	F [♮]	F [♮]	G [♮]	.	A [♮]
D [♮]	E	E [♮]	F [♮]	F [♮]	G [♮]	.	A	A [♮]	B	B [♮]	C [♮]	C [♮]	D [♮]
G [♮]	A	A [♮]	B	B [♮]	C [♮]	C [♮]	D	D [♮]	E	E [♮]	F [♮]	F [♮]	G [♮]
C [♮]	D	D [♮]	E	E [♮]	F [♮]	F [♮]	G	G [♮]	A	A [♮]	B	B [♮]	C [♮]
F [♮]	G	G [♮]	A	A [♮]	B	B [♮]	C	C [♮]	D	D [♮]	E	E [♮]	F [♮]
B	C	C [♮]	D	D [♮]	E	E [♮]	F	F [♮]	G	G [♮]	A	A [♮]	B
E	F	F [♮]	G	G [♮]	A	A [♮]	B [♮]	B	C	C [♮]	D	D [♮]	E
A	B [♮]	B	C	C [♮]	D	D [♮]	E [♮]	E	F	F [♮]	G	G [♮]	A
D	E [♮]	E	F	F [♮]	G	G [♮]	A [♮]	A	B [♮]	B	C	C [♮]	D
G	A [♮]	A	B [♮]	B	C	C [♮]	D [♮]	D	E [♮]	E	F	F [♮]	G
C	D [♮]	D	E [♮]	E	F	F [♮]	G [♮]	G	A [♮]	A	B [♮]	B	C
F	G [♮]	G	A [♮]	A	B [♮]	B	C [♮]	C	D [♮]	D	E [♮]	E	F
B [♮]	C [♮]	C	D [♮]	D	E [♮]	E	F [♮]	F	G [♮]	G	A [♮]	A	B [♮]
E [♮]	F [♮]	F	G [♮]	G	A [♮]	A	B [♮]	B [♮]	C [♮]	C	D [♮]	D	E [♮]
A [♮]	B [♮]	B [♮]	C [♮]	C	D [♮]	D	.	E [♮]	F [♮]	F	G [♮]	G	A [♮]
D [♮]	.	E [♮]	F [♮]	F	G [♮]	G	.	A [♮]	B [♮]	B [♮]	C [♮]	C	D [♮]
G [♮]	.	A [♮]	B [♮]	B [♮]	C [♮]	C	.	D [♮]	.	E [♮]	F [♮]	F	G [♮]
C [♮]	.	D [♮]	.	E [♮]	F [♮]	F	.	G [♮]	.	A [♮]	B [♮]	B [♮]	C [♮]
F [♮]	.	G [♮]	.	A [♮]	B [♮]	B [♮]	.	C [♮]	.	D [♮]	.	E [♮]	F [♮]
B [♮]	.	C [♮]	.	D [♮]	.	E [♮]	.	F [♮]	.	G [♮]	.	A [♮]	B [♮]
L	1	L	1	L	1	D	1	L	1	L	1	L	1

Tab.
of beats in 15 Seconds, of

Pl. XX.

33	49	36	27	32	47	35	53
32	47	35	26	31	46	34	52
31	46	34	26	30	45	34	50
30	45	34	25	29	44	33	48
29	44	33	24	28	42	32	47
28	42	32	24	28	41	31	46

o	bo				*o		*o
	bo		o		*o		*o
o		bo		o		*o	
			o				

Tab.
of beats in 15 Seconds

29	44	33	24	29	43	32	48
28	43	32	24	28	42	31	47
28	41	31	23	27	40	30	45
27	40	30	23	26	39	29	44
26	39	29	22	26	38	29	43
25	38	28	21	25	37	28	42

o	bo				*o		*o
	bo		o		*o		*o
o		bo		o		*o	
			o				

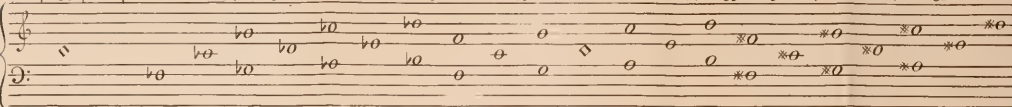
Tab. v.

o		o		o	*o		*o
		o		o	*o		*o
bo		o		o		*o	
			o				

Tab. II

The numbers of beats in 15 Seconds, of the v^{th} in the System of equal harmony.

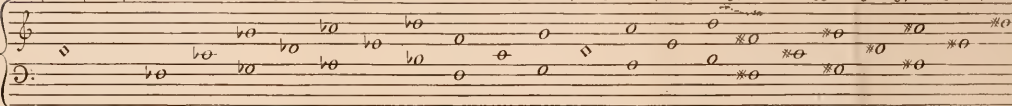
Pitch. $\frac{1}{4}$ tones																					
293	4	26	38	29	43	33	49	36	27	41	30	45	34	50	38	28	42	32	47	35	53
285	3	26	38	28	41	32	47	35	26	40	30	44	33	49	37	28	41	31	46	34	52
277	2	25	37	28	40	31	46	34	26	39	29	43	32	48	36	27	40	30	45	34	50
269	1	24	36	27	39	30	45	34	25	37	28	42	31	47	35	26	39	29	44	33	48
262	0	23	35	26	38	29	44	33	24	36	27	41	30	45	34	25	38	28	42	32	47
255	1	23	34	25	37	28	42	32	24	35	26	40	30	44	33	25	37	28	41	31	46



Tab. I

The numbers of beats in 15 Seconds, of the v^{th} in the System of mean tones.

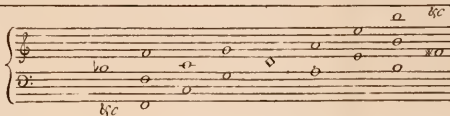
Pitch. $\frac{1}{4}$ tones																					
293	4	23	35	26	39	29	44	33	24	37	27	41	30	46	34	26	38	29	43	32	48
285	3	23	34	25	38	28	43	32	24	36	27	40	30	45	33	25	37	28	42	31	47
277	2	22	33	25	37	28	41	31	23	35	26	39	29	43	32	24	36	27	40	30	45
269	1	22	32	24	36	27	40	30	23	34	25	38	28	42	31	24	35	26	39	29	44
262	0	21	31	23	35	26	39	29	22	33	24	37	27	41	31	23	34	26	38	29	43
255	1	20	30	23	34	25	38	28	21	32	24	36	27	40	30	22	33	25	37	28	42



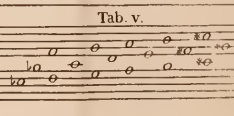
Tab. iv.



Tab. iii.



Tab. v.



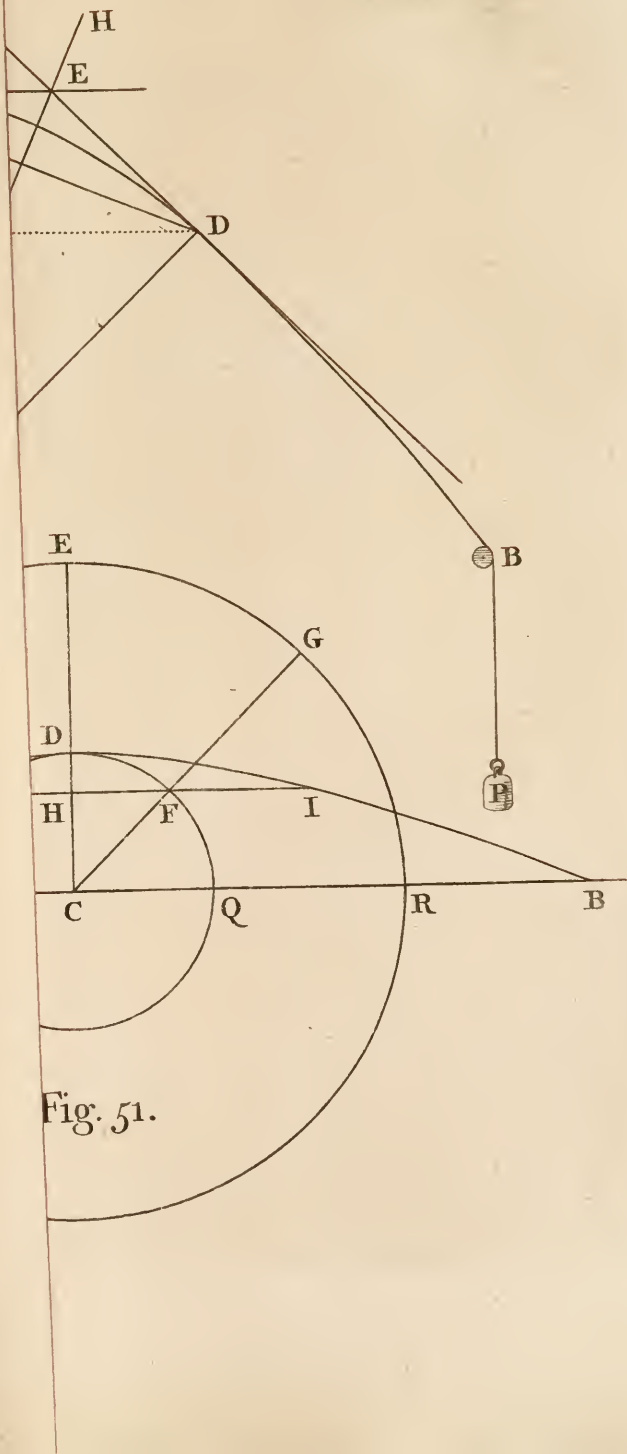


Fig. 51.



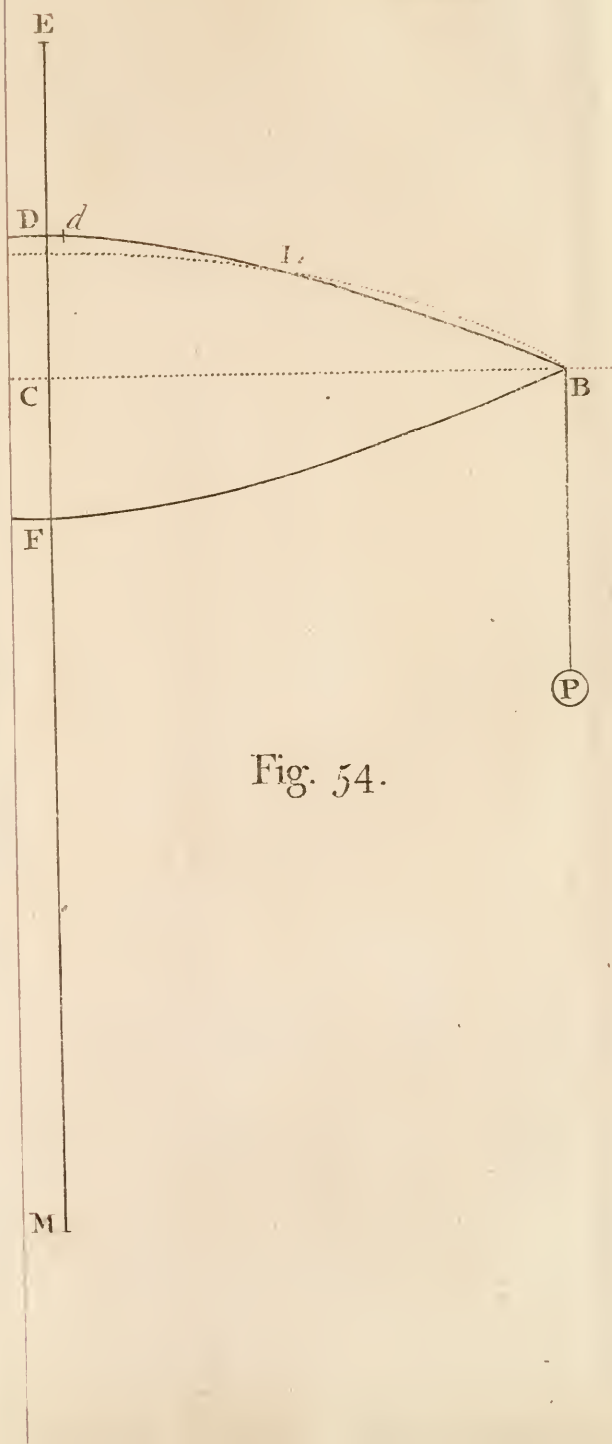


Fig. 54.

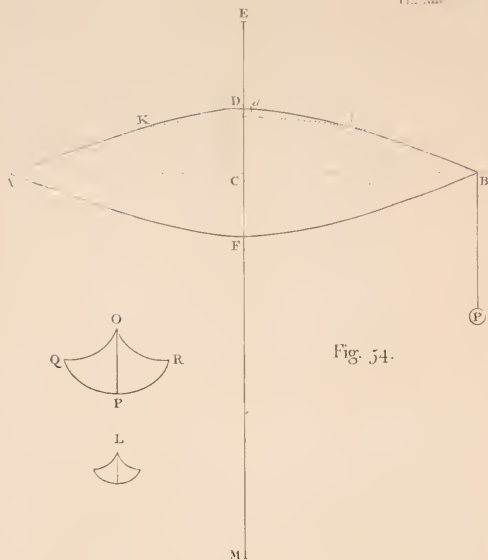
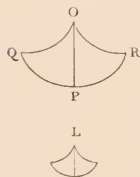


Fig. 54.



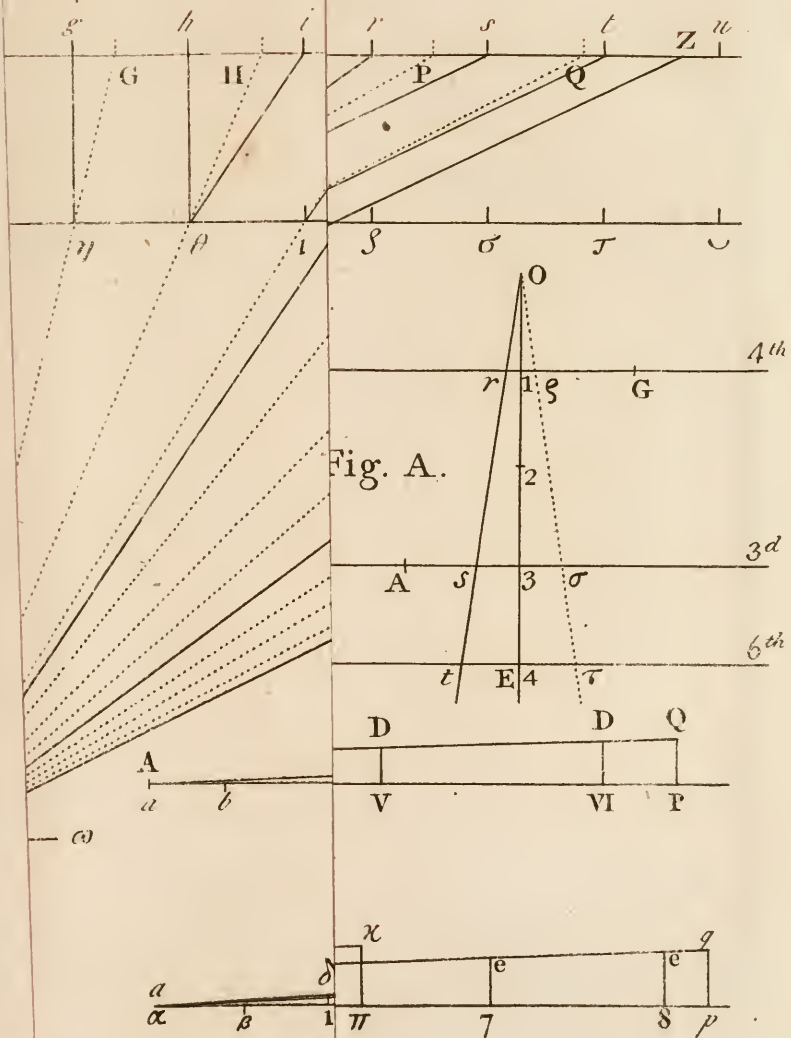


Fig 56

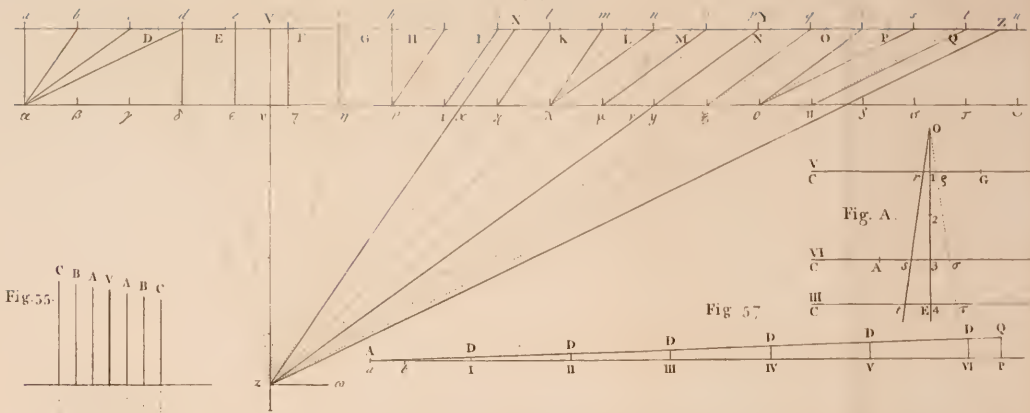


Fig. 55.

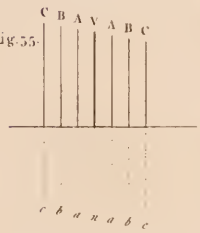


Fig 57

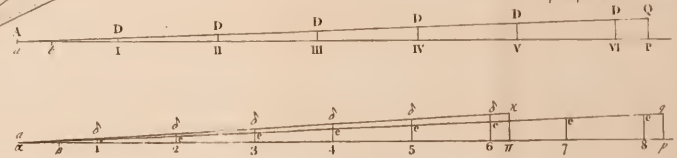


Fig. 58.

PL. XXV.

G D G
g e g

G F F D E F G
g f d e f g

L M N W X Y
l m n n x y

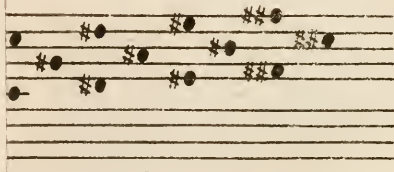
L Y
l n n y

G N X
m x

G M W
l n n y

G K
i

i



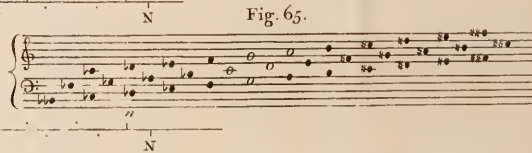
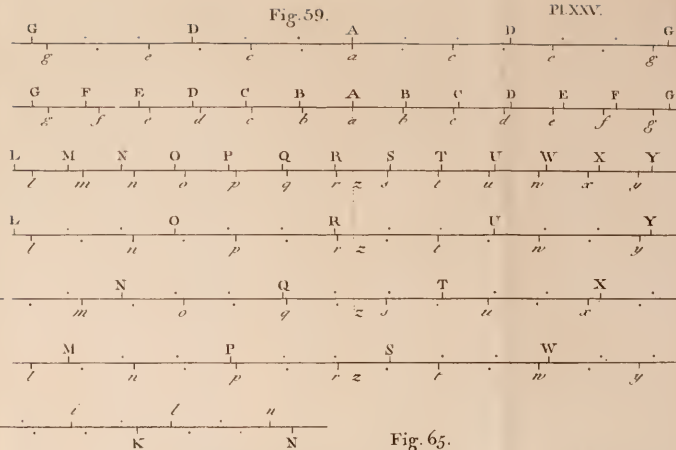
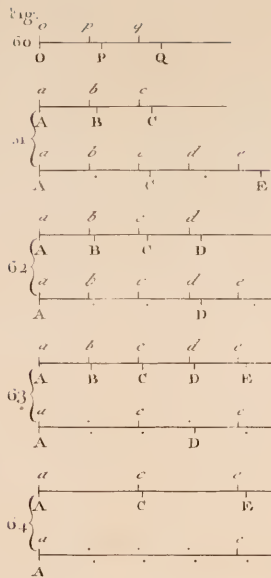
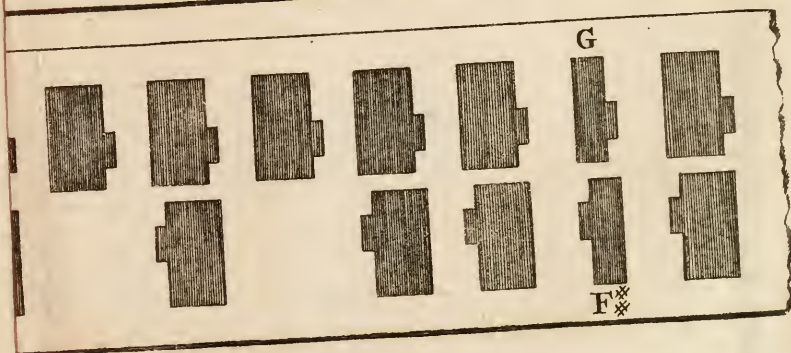
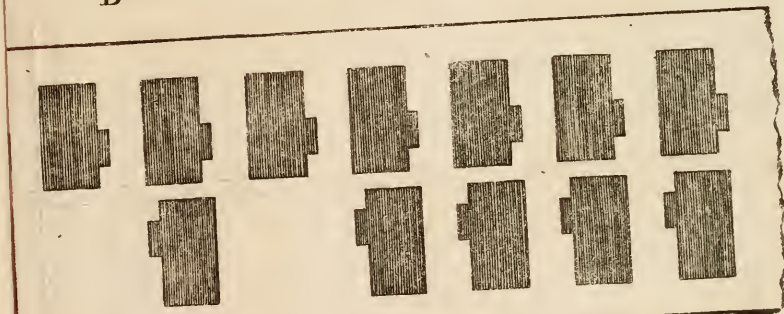
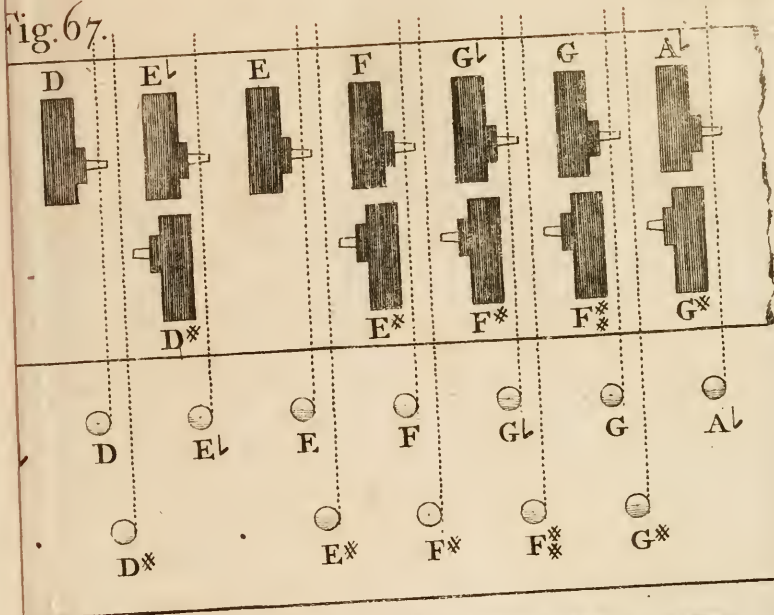


Fig. 67.



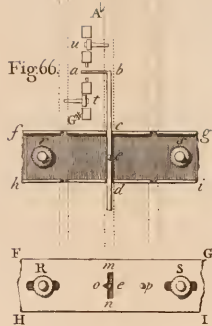
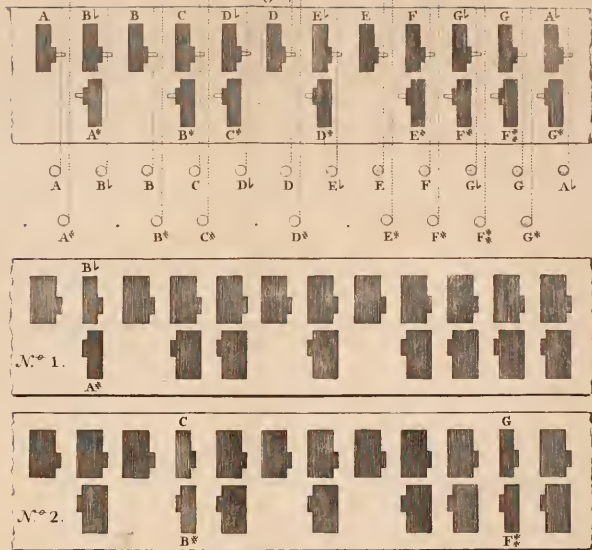
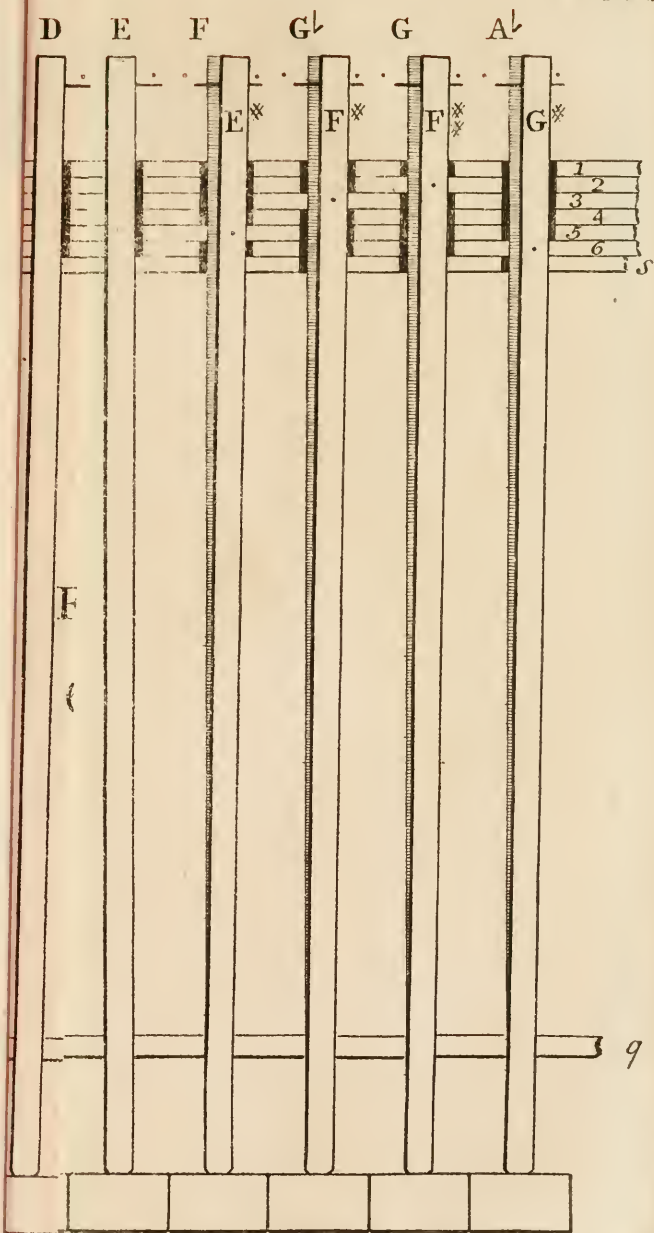
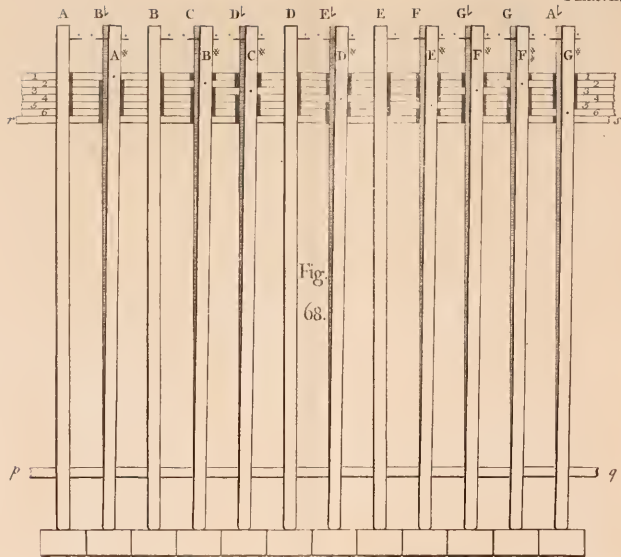


Fig. 67.

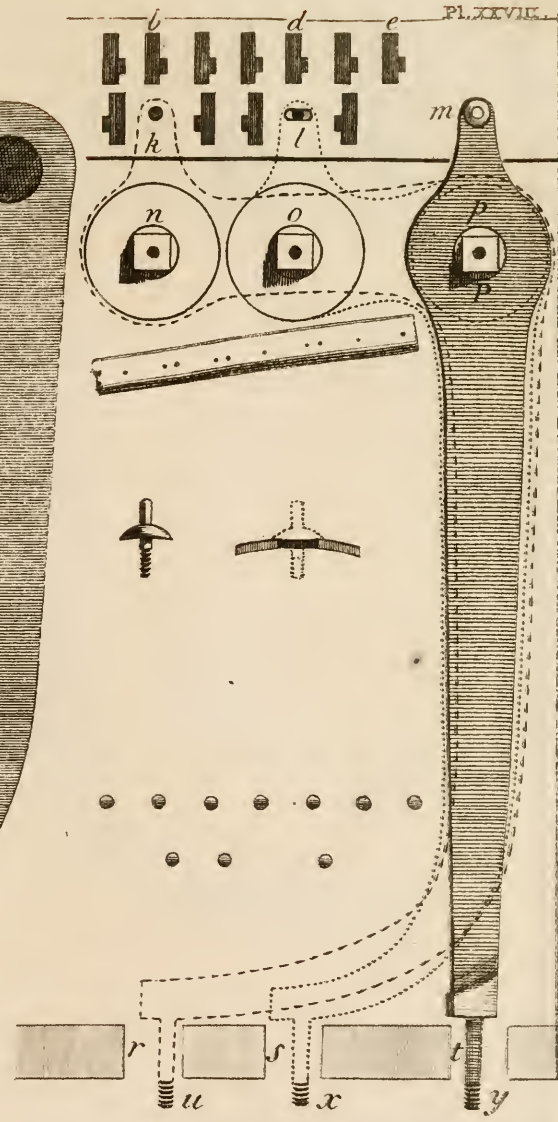
PL. XXXI.



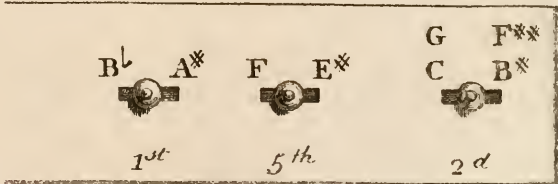




69.



b. 70.



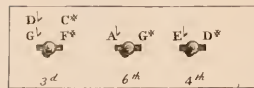
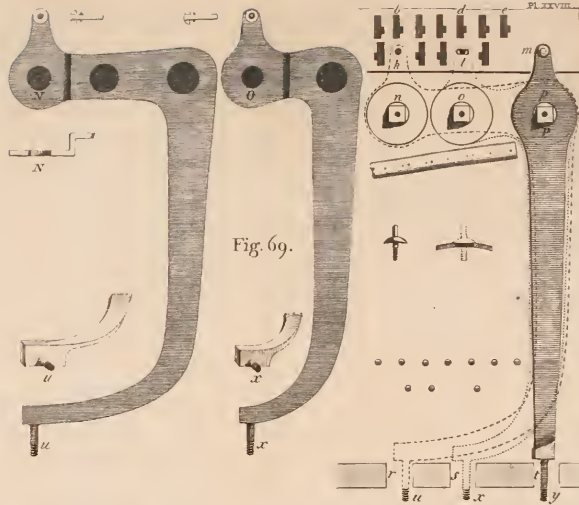
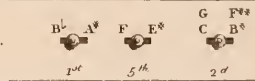
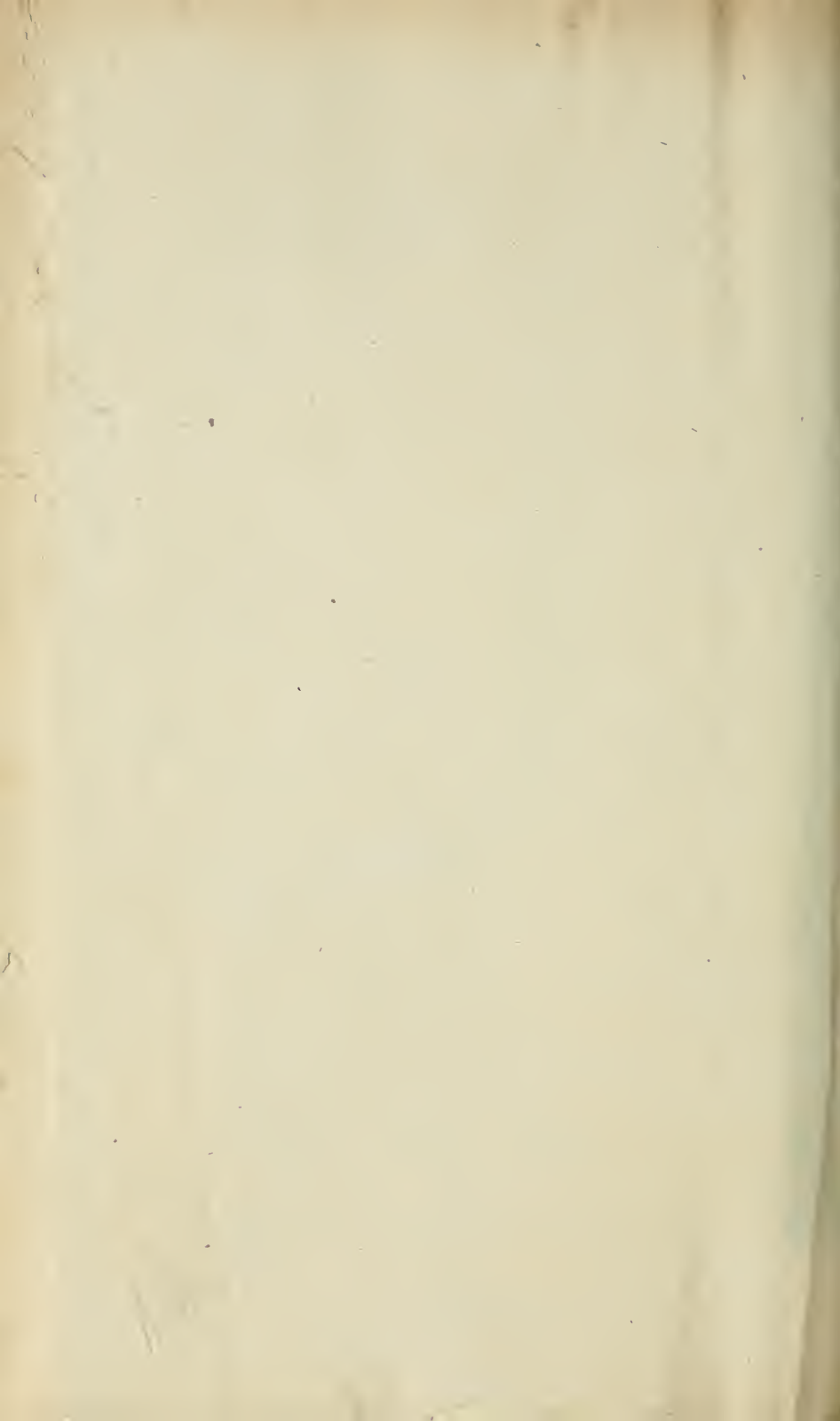
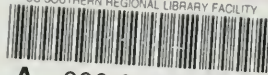


Fig. 70.





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